Opportunities and Alternatives for Training Students in Using Mathematical Proof in Primary School

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Abstract

Mastering logical proofs in primary school is a prerequisite for the development of mathematical thinking in students. They are an important alternative route for the utilization of under-used mental capacities of students. Particularly valuable in this field is the use of indirect apagogical proof.

The aim of this publication is to present our research on the theoretical foundations of the indirect apagogical proof and the methodological possibilities for studying this topic in primary mathematics education. This study is also aimed at both discovering and presenting some possibilities, offered by the primary mathematics curriculum in Bulgaria, for propaedeutic training in using the apagogical proof.

The theoretical aspects of this work reveal the complex logical structure of the proof in question.

The developed methodology determines how training of primary school students is done through a system of principles, methods and means of training.

As a result of the training, after the implementation of the developed technological option for training in this field, we expect students to have mastered the knowledge and skills to solve problems by applying the indirect apagogical proof, in accordance with their age and perceptive abilities.

1. Introduction

The study of logical proofs in primary school is not part of the mandatory educational mathematical content in Bulgaria, but it is a virtuoso teaching tool that contributes to the formation and development of students' logical and mathematical thinking. This paper focuses on the indirect apagogical proof.

2. Method

The purpose of this publication is to present our research on the theoretical foundations of the indirect apagogical proof and the methodological possibilities for the study of the subject in the initial training in mathematics. This study is also aimed at the detection and presentation of some possibilities that are offered by the educational mathematical content during the primary education in Bulgarian schools for propaedeutic training in indirect apagogical proof.

2.1. Theoretical framework

The conduct of an indirect apagogical proof is implemented through the following three stages:

First stage. Formulation of logical negation $\bar{T}$ of the statement T, the precision of which should be proven.

Second stage. Establishing the incorrectness of the formulated negation by the Euclid's Algorithm, i.e. number of consequences are derived from $\bar{T}$. The last of these consequences has to be false. From its incorrectness comes the incorrectness of $\bar{T}$.

Third stage. From the incorrectness of $\bar{T}$, on the basis of the law of the excluded middle, a conclusion is made on the correctness of T.

The complex logical structure of the proof in question determines the need for thorough preparation for its assimilation.

In the following exposition we will discuss the key points from the assimilation technology of the stages of the indirect apagogical proof, using mainly the possibilities of the curricula in mathematics in the Bulgarian primary school.

The first stage of the method is connected with the logical operation „negation of proposition" and formulation of contradictory proposition of a certain proposition. Therefore, the primary focus of the exercise here is related to the proper formulation of a contradictory proposition of a certain proposition. The attention of the learners is drawn to the fact: only one of two contradictory propositions is true.

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1) Formulation of negation of a single proposition
If p is a single proposition, its contradictory proposition is $\bar{p}$ is „not-p“. If p is a single affirmative (negative) proposition, then $\bar{p}$ is a single negative (affirmative) proposition.
Here is an example of a pair of contradictory propositions:
p: The number 36 is divided by the number 4.
$\bar{p}$: The number 36 is not divided by the number 4.
The students easily assimilate the pairs of propositions of the concerned type and determine their correct values.
2) Formulation of contradictory propositions from the logical square
The following pairs of propositions are contradictory: universal affirmative (SAP) and particular negative (SOP); universal negative (SEP) and particular affirmative (SIP).
There is a need of clarification of the meaning and the importance of the quantifiers of community and of existence, which are used in the propositions of the logical square; as well as the fact that the negation of a proposition of community is a proposition of existence and the negation of the proposition of existence is a proposition of community.
The attention of the learners is drawn to the wording and the juxtaposition of the contradictory propositions from the logical square of propositions.
We will cite examples of pairs of contradictory propositions from the logical square, which can be used in the learning process of mathematics.
- ✓ SAP: All squares are rectangles.
  SOP: Some squares are not rectangles.
- ✓ SIP: Some numbers are odd.
  SEP: All numbers are not odd.

3) Formulation of negation of compound proposition
4) Differentiation of contradictory from opposite propositions
The propositions of the types SAP and SEP are opposite. They cannot both be true, but they can both be incorrect.
This can be seen from the following examples:
- ✓ SAP: All equilateral triangles are isosceles.
  SEP: All equilateral triangles are not isosceles.
  SOP: Some equilateral triangles are not isosceles.
- ✓ SEP: All numbers are not divided by two.
  SAP: All numbers are divided by two.
  SIP: Some numbers are divided by two.

For all three examples the first two statements are opposite, and the first and the third – contradictory.
In the second stage of the negation method is used the rule of denial, which in the mathematical logic is written as follows:

(1) $$\frac{p \to q, \bar{q}}{\bar{p}},$$

where p and q are propositions, and $\bar{p}$ and $\bar{q}$ are their logical negations.
We will illustrate the application of this rule with the solving of the following problems from the mathematical curriculum:
Problem 1. Is it true that 42:7=8?
We will present the solution in a schematic type (1):

$$\frac{42:7=8 \to 8.7=42, 8.7 \neq 42}{42:7 \neq 8}.$$

Here we use the interrelationship between division and multiplication.
The following must be explained to the students: $42:7 \neq 8$, because $8.7 \neq 42$.
Problem 2. Is it true that for every number the following has been fulfilled: $a.a = a$?
The solution of this problem also can be presented with the type (1):

$$\frac{\forall a : a.a = a \xrightarrow{a=2} 2.2 = 2, 2.2 \neq 2}{\overline{\forall a : a.a = a}}.$$

In the third stage of the negation method is applied the law of the excluded middle, according to which one of two contradictory propositions is necessarily true and there is no possibility of a third true proposition.
The attention of the learners is drawn to the fact that: the elements of a given set are divided into two disjoint sets, depending on whether they have a certain sign or not; there is no third subset.
This can be accomplished by the following types of problems:
Which of the numbers 3;10;115;81: a) can be divided by 3; b) are double digits?
Thus, through targeted and accessible exercises the learners can prepare for the assimilation of the stages of the negation method on the basis of the educational content in mathematics.

2.2. Technological option for study of the indirect apagogical proof

This proof is mostly applied to problems, in which must be proven that there is no element of the set P, which to satisfy the condition L, or in which must be proven that every element of the set P satisfies the condition L.
The present exposition focuses on the design of a theoretical framework of the indirect apagogical proof in specific problems.
We will present methodological guidelines for some of the examples.
Example 1. The numerical value of the expression 42.6–498:*x* is smaller than the numerical value of the expression 28.*x*+2. Prove that the number *x* does not equal the numerical value of the expression 24–3.7.
Reasoning:
In this example prove the statement T: The number *x* does not equal 24–3.7.
① We formulate the negation of the statement of the problem
$\bar{T}$: Let *x*=24–3.7, i.e. *x*=3.
② We come to the consequences:
T_1: 42.6–498:3<28.3+2;
T_2: 252–166<84+2;
T_3: 86<86.
The statement T_3 is not correct.
Therefore, $\bar{T}$ is not correct.
③ From the incorrectness of $\bar{T}$ comes the correctness of the statement, formulated in the problem.
Example 2. The numerical value of the expression (*a*–1080):15+*b* is bigger than the difference of the numbers 1250 and 250. Prove that *b* does not equal 729, if *a*=5145.
Example 3. Twelve football players are lined up in a row. The coach tells them that every two players, divided by another one, can change places. Prove that no matter how many changes take place, the players will never be able to arrange themselves in an order which is reverse to the original.
Reasoning:
① We formulate the negation of the statement of the problem:
$\bar{T}$: The football players can be arranged in an order which is reverse to the original.
② We come to the consequences:
T_1: The football player, who was first, will go to the twelfth place; the one, who was second, will go to the eleventh place and so on.
T_2: Each football player at an odd place goes to an even place, and from an even place goes to an odd place.
The statement T_2 cannot be true, because the parity of the occupied place does not change with the shift of every other person.
Therefore, $\bar{T}$ is not correct.
③ From the incorrectness of $\bar{T}$ comes the correctness of the statement, formulated in the problem.
Example 4. Thirteen students participate in an all-against-all type of chess tournament. Prove that it is not possible that at a certain moment each of them has participated in a total of five games.
Example 5. One of the numbers from 1 to 6 (including) is written to each side of a hexagon in random order, without repeating. Similarly, the same numbers are written to the segments joining the inner point of the hexagon with its vertices. Prove that there is no such arrangement of the numbers in the

specified manner, in which all sums of the triad numbers written to the sides of the resulting triangles are equal.
Example 6. Ivo has four laths with lengths over 1 m, two laths with a length of 2 m and one lath with a length of 3 m. Prove that Ivo cannot enclose a rectangular section with all the laths without cutting them.
Reasoning:
① We formulate the negation of the statement of the problem:
$\bar{T}$: Ivo could enclose a rectangular section with the given laths (without cutting them).
② We come to the consequences:
T_1: $4.1 + 2.2 + 1.3 = 11$ (m is the circumference of the rectangle);
T_2: The resulting number is odd.
This statement cannot be true, because the circumference of the rectangle is an even number.
Therefore, $\bar{T}$ is not correct.
③ From the incorrectness of $\bar{T}$ comes the correctness of the statement T.
Example 7. Emma has one sheet of paper. She cuts it into four parts. Then she cuts some of the new parts into four parts and so on. Prove that in this way she cannot receive exactly 50 parts.
Reasoning:
① We formulate the negation of the statement of the problem:
$\bar{T}$: Emma can have exactly 50 parts in the specified way.
② We come to the consequences:
T_1: With each cutting the number of pieces increases with 3.
T_2: The number of pieces obtained in this way is $M = 3.n + 1, n \in N$.
But the statement T_2 cannot be correct, because when divided by 3 the number 50 does not have 1 as a residual.
Therefore, $\bar{T}$ is not correct.
③ From the incorrectness of $\bar{T}$ comes the correctness of the statement, formulated in the problem.
Example 8. The numbers 2;7;9;10;3;13 are written sequentially on the vertices of a hexagon. We can choose two adjacent vertices and simultaneously to increase or reduce the numbers in them with the same number. Can a few such actions in the vertices result in the numbers 5;11;6;15;7;14?[1]
Example 9. We are given a line of 1000 numbers, marked with $A_1, A_2, A_3, \ldots, A_{999}, A_{1000}$. We are allowed to change the locations of any two numbers, between which there are exactly 5 other numbers. Can we get the line $A_{1000}, A_{999}, \ldots, A_3, A_2, A_1$ only with such change? [2]

3. Findings
The structure and the content of this exposition formed one option of training of students from the primary education. This option represents a complete training resource. Its application can be implemented during the optional courses, the mandatory elective training in mathematics, as well as in mathematical schools and workshops.
Upon the design of the present educational resource in the mathematical education, we can present the following as the expected results from the training of the students:

- Assimilation of the structure of the indirect apagogical proof.
- Formation of a set of skills and strategies to solve problems by using indirect apagogical proof.
- Development of logical thinking.

4. Conclusion
The presented possibilities and mathematical ideas to solve problems by using indirect apagogical proof are an alternative to the traditional content and if we apply them when teaching mathematics they can facilitate the development of the logical thinking of the students.

References
[1] Paskaleva, Z. et al. (2008*) Tests in mathematics and tasks with fun and logical nature of decisions*. Sofia:Arhimed.
[2] Zlatilov, V. et al.(2005) *Math reader book for children*. Sofia:Trud&Prozorets.