Enhancing Mathematical Understanding through an Optimization Problem of Delay at Fixed-Cycle Traffic Lights

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Abstract

Delay at traffic lights is a real life problem that each of us has ever encountered. Therefore it is an ideal example for the promotion of the applicability and usefulness of mathematics. As delay is to be avoided, an optimization problem arises as a function of the time period of green lights. The function to be minimized is also another interesting discussion item. Queues at fixed-cycle traffic lights are mostly studied from the point of view of an individual driver, but as the time period of green lights coincides with the time period of red lights in the perpendicular direction, the interests of the different drivers in those directions are opposite. The more advanced modeling of the general delay at an intersection depends on the arrival rates in each of the directions and the departure rate. The model is the basis to find an optimal tuning of the red and green periods at the fixed-cycle traffic lights. The Webster's [1] and Newell's [2] delay formula's are studied and discussed for varying arrival rates in the perpendicular directions. Restrictive assumptions are motivated and handled. The delay model is useful in mathematical teaching as only elementary knowledge is a prerequisite. A detailed analysis offers easy progression in comprehension of the influences of each of the determining parameters. This example can go beyond regular high school mathematics and reaches a deeper beleef in the usability of mathematics. Visualizations with software (Maple, Matlab...) give an additional value to select an optimal tuning of the traffic lights. Classroom activities are presented ready for use.

1. Introduction

We consider the traffic flow at signalized isolated intersections where queues arise at a junction of two roads with fixed-cycle traffic lights (FCTL). An isolated intersection is formed at the intersection of two streets that can each carry two opposite flows of vehicles. As depicted in Fig.1, we restrict our discussion to a one-way to one-way intersection. With no loss of generality, we take the direction of the flow in the first street northwards. The other street conducts a one-way eastward flow. Cars arrive at the south and west entrances of the intersection.
The following assumptions are made for the FCTL queues discussed in this paper [4]:

- The queue built up during the red lights period, is cleared at the end of a cycle of length C.
- The queue has no overflow at the end of a cycle of length C.
- The number of vehicles that arrive at the intersection is independent of the slot or cycle.

The traffic lights alternate between green and red, where the queue is build up during the period of length R of red lights, while the queue will clear during the period of length G=C-R of green lights with C the fixed cycle length. The mean delay has been studied by several authors: Webster [3] deduced a formula from simulations to express the mean delay of a vehicle in a FCTL queue. As most of the research on FCTL, it is based on Poisson arrivals. For those who are based on another distribution, the assumption is made that the arrivals occur independently.
Van den Broek and van Leeuwaarden [4] [5] already described bounds and performance characteristics for delay at FCTL. In this paper we study the optimal value for G to minimize the total delay at the intersection. We deal with the problem from a general point of view rather than from an individual angle of a single driver. Like that we try to find an optimal tuning of the traffic lights where an average driver (in the north-south direction as well as in the east-west direction) will is subject to the least delay. An analysis is described in section 2 based on well known formulas from literature [1] [2] [3]. Section 3 discusses the way this real life example can be used in the classroom

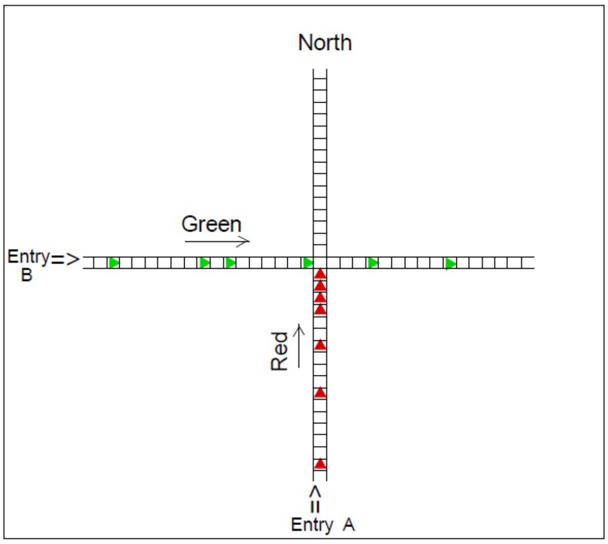

Fig. 1 An isolated intersection.

2. Webster's and Newell's formula

The mean delay associated with FCTL queues with Poisson arrivals, was modeled by Webster [3] as

$$E[D_W] = \frac{(C-G)^2}{2\,C\,(1-\rho)} + \frac{\rho\,C^2}{2\,G\,(G-\rho\,C)} - 0.65\left(\frac{C}{\rho^2}\right)^{\frac{1}{3}} \left(\frac{\rho\,C}{G}\right)^{2+5\,G/C}$$

with ρ the arrival rate of vehicles.

The first term in this formula represents delay when traffic can be considered arriving at a uniform rate, while the second term represents the random nature of the arrivals assuming a Poisson arrival process with mean value ρ. Following the first two terms the departures are supposed to occur at constant rate $\rho\,C/G$ which corresponds to the signal capacity. As this is not in correspondence to the assumption of a realistic situation, the last term is added as a correction term, found out of simulation results. This has become a standard formula, while many refinements have been made, sometimes under certain secondary conditions. Newell [1] included the variance of the arrival process in his formula for the mean delay:

$$E[D_N] = \frac{(C-G)^2}{2\,C\,(1-\rho/\mu)} + \frac{(C-G)I}{2\,C\,\mu\,(1-\rho/\mu)^2}$$

with μ the service time or departure flow rate from the queue during the period of length G (vehicles/second) and ρ the arrival flow rate (vehicles/second). Assuming there is no overflow means that $\mu\,G < \rho$ is supposed.

The index of dispersion I is defined as

I=variance of the number of arrivals per cycle / mean number of arrivals per cycle.

3. Optimization of the mean delay

When we construct an average delay function $D_i(G,\rho) = E[D_i]$, $i\epsilon\{W,N\}$, we can define the total delay at an intersection as described in Section 1, defined as

$$D_{total,i} = D_i\,(\,G\,\rho_1 + C - G\,\rho_2), i\,\epsilon\{W,N\}.$$

In case of a cycle length of two minutes, a fixed arrival rate ρ_1 in one direction and ρ_2 in the opposite direction, the total delay behaves as a function of G as in Fig.2 and Fig.3, which makes it possible to determine the suited G-value to reach a minimum for $D_{total,i}$. Fig.4 shows this optimal G-value for

varying arrival rates, I=0.5, C=120, $\mu = 0.7$ following Webster and Newell. It makes clear that under the circumstances of the example, Newell's optimal G-values are larger then Webster's as they diverge more from the central value of 60 in a two minutes cycle for similar arrival rates, even if the variation coefficient I=0.5 is small. The influence of the variance value I is clear from Fig. 5, where for two values of I the optimal G value is shown for varying values of arrival rates $\rho_1$and ρ_1 and fixed departure rate $\mu = 0.7$. In these conditions it is shown that the smaller I, the closer the optimal is situated in the neighborhood of the central value 60.
These conclusions are confirmed in Table 1 and Table 2. The last table also shows that when the departure rate is larger, there is less need to deviate the optimal value G_{opt} from the central value 60.

ρ_1	ρ_2	G_opt
0.2	0.2	60
0.2	0.4	41.66
0.4	0.2	78.33
0.5	0.2	93.21

Table 1. Some optimal G_{opt} values under varying circumstances to optimize the flow at the two-road intersection following Webster's delay concept.

ρ_1	ρ_2	μ	I	G_opt
0.2	0.2	0.7	0.5	60
0.4	0.2	0.7	0.5	75.33
0.4	0.2	0.7	5	78.33
0.5	0.2	0.7	0.5	86.46
0.5	0.2	0.7	5	93.21
0.4	0.2	1.4	0.5	65.45
0.4	0.2	1.4	5	65.87

Table 2. Some optimal G_{opt} values under varying circumstances to optimize the flow at the two-road intersection following Newell's delay concept.

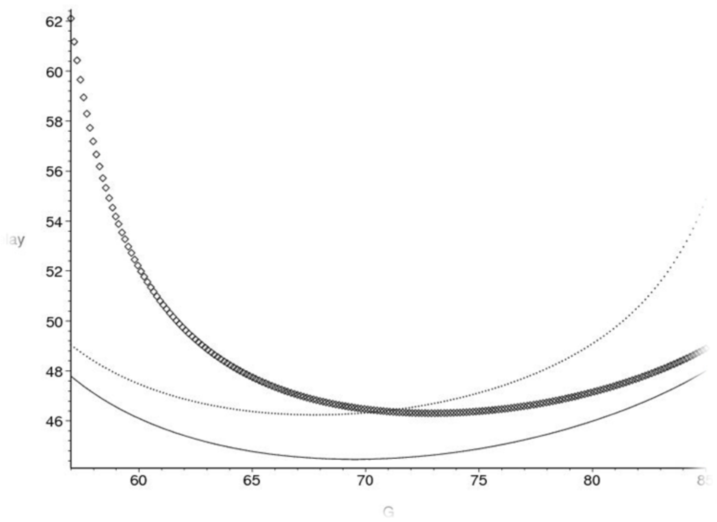

Fig.2 The average total delay $E[D_{total,W}]$ at a two road junction with cycle length C=120 and fixed arrival rates $\rho_1 = 0.4$ and $\rho_2 = 0.2$ (full line) / $\rho_1 = 0.4$ and $\rho_2 = 0.25$ (dots) / $\rho_1 = 0.45$ and $\rho_2 = 0.2$ (diamond).

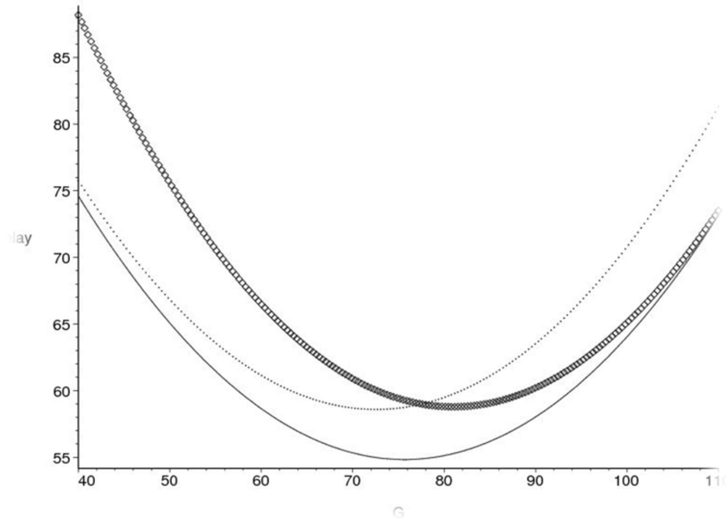

Fig.3 The average total delay $E[D_{total,N}]$ at a two road junction with cycle length C=120, departure rate $\mu = 0.7$, variance I=1 and fixed arrival rates $\rho_1 = 0.4$ and $\rho_2 = 0.2$ (full line) / $\rho_1 = 0.4$ and $\rho_2 = 0.25$ (dots) / $\rho_1 = 0.45$ and $\rho_2 = 0.2$ (diamond).

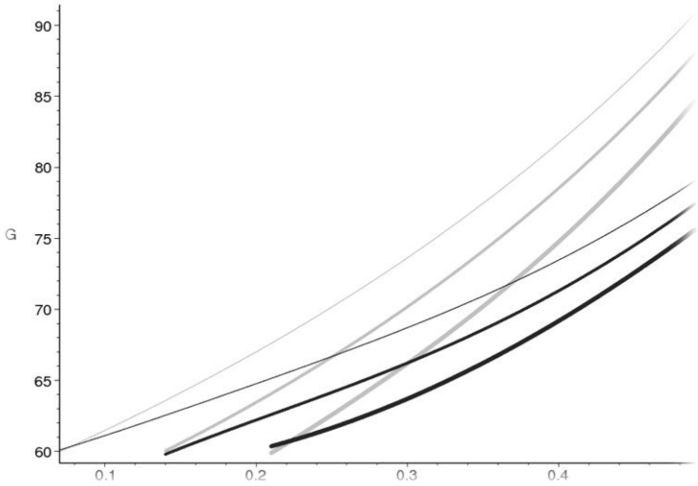

Fig.4 The value for G to optimize the mean total delay $E[D_{total,N}]$ (grey) and $E[D_{total,W}]$ in case of I=0.5 as a function of ρ_1 for ρ_2 =k 0.07 (k=1,2,3, thicker line for increasing ρ_2).

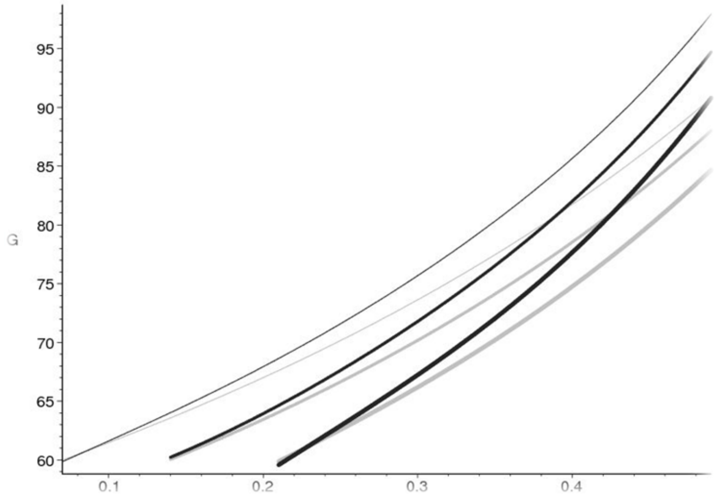

Fig.5 The value for G to optimize the mean total delay $E[D_{total,N}]$ in case of I=0.5 (grey) and I=5 (black) as a function of ρ_1 for ρ_2 =k 0.07 (k=1,2,3, thicker line for increasing ρ_2).

4. Conclusions

This topic of traffic light time optimization shows the students that individual optimization differs from an optimization where the intersection is considered as an entire unit. As a profitable decision in terms of green time in one direction is disadvantageous for the perpendicular direction, it is a challenge to find a global optimum. From this global point of view we suggested a way to find an optimal partition of the time the traffic lights are green and red respectively at a two-road intersection. Students can be asked to generate each of the figures by themselves with mathematical software. Discussions about the contents of the figures can start from questions as

- Are you surprised about the position of the different curves in the figure? Can you explain their position?
- What is the influence of the variance parameter I?
- Do you recognize the trivial results of G=R for equal arrival rates in both directions out of the more general approach for arbitrary arrival rates?
- Why does the outcome increase/decrease as …. increases/decreases?

The resulting discussion will end up in a comprehension of the influences of each of the determining parameters.

This subject was presented to our first year engineering students. They witnessed to be surprised of the usability of mathematics. They appreceated the figures as they make the formula come alive.

References

[1] Newell, G.F. (1960). Queues for a fixed-cycle traffic light. Ann. Math. Statist. **31** p. 589-597.

[2] Newell, G.F. (1965). Approximation methods for queues with applications to the fixed-cycle traffic lights. SIAM Rev. **7**, p. 223-240.

[3] Webster, F.V. (1958). Traffic signal settings.} Road Res. Laboratory Tech. Rep. **39**.

[4] Van den Broek, M.S., van Leeuwaarden, J.S., Adan, I.J., Boxma, O.J. (2006). Bounds and Approximations for the Fixed-Cycle Traffic-Light Queue.

[5] Van Leeuwaarden, J.S. (2006). {\em Delay analysis for the fixed-cycle traffic-light queue} Transportation Sciences **40(4),** p. 484-496.