

## Learning about Probability through Game Strategies

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### Abstract

*We present a mathematical activity that can be used to reach a deeper understanding of probability for students of grade 8-14. By means of dice games teachers can introduce students to the concepts of probability through hands-on exercises that can be implemented in the classroom. The described approach is one of an active learning environment, a platform to engage students in the learning process and may increase student/student and student/instructor interaction. The activities require use of some basic summation results, binomial coefficients and limits. As such they build on knowledge students have acquired in prior classes. The game we discuss, requires a stake of both players. Determining the winner is only a matter of luck as it is based on a die. But the reward after winning is related to the stakes chosen by the players. So the question arises which stakes lead to a fair game, i.e. the average net profit for both players is zero. The material described in this paper can be used by faculty members teaching probability at the undergraduate level. Student activities are presented here student can use mathematics as a tool to discover the unfairness and to design a better strategy to make the game fair.*

### 1. Introduction

By means of dice games teachers can introduce students to the concepts of probability through hands-on exercises that can be implemented in the classroom. The described approach is one of an active learning environment, a platform to engage students in the learning process and may increase student/student and student/instructor interaction. The activities require the use of some basic summation results, binomial coefficients and limits. As such they build on knowledge students have acquired in prior classes. The game we will discuss requires a stake of both players. Determining the winner is only a matter of luck as it is based on a die. But the reward after winning is related to the stakes chosen by the players. So the question arises which stakes lead to a fair game, i.e. the average net profit for both players is zero.

### 2. The basic game

#### 2.1. Equal chance of winning

Consider a game based on the following simple rules: two players stake individually an amount of money. Then both players toss a die. The highest result indicates the winner who gets both stakes. The first point of understanding that should be reached with our students is that if the value of the stake is different for both players, the risk they take is different, so the probability of winning should be different proportionally. This is the main aim of student exercise 1. Let  $x_B$  be the stake of player B and  $x_A$  the stake of player A. The following student exercises require little theoretical background and are based on logical thinking. Previous lecture material will suffice, so any guidance from a teacher will be needless.

Student exercise 1: *Simulate the game by-hand by forming groups of two players and discuss whether the game seems fair. Do this in case of*

(a)  $x_A \in [0, 5]$  and  $x_B \in [3, 8]$

(b)  $x_A \in [0, 5]$  and  $x_B \in [0, 5]$

#### 2.2. Adapted strategies

Suppose our game is a more exotic variant of the basic game, based on one of the following ways to increase the probability of winning for player B: player A tosses the die only once with result  $y_A$  while player B tosses twice giving the two values  $y_1$  and  $y_2$ . The final result of player B can be defined as:

- $y_B = \max(y_1, y_2)$
- $y_B = y_1 + y_2$
- $y_B = \frac{y_1 + y_2}{2}$

Player A wins if  $y_A > y_B$  and vice versa. The game stays undecided if  $y_A = y_B$ . The modified decision rules will give a game where on average A will not get richer if  $x_A = x_B$ , as B has the largest probability of winning. Here we feel the necessity to investigate which stakes lead to a game where the net profit for both players is zero. Therefore we will use the factor

$$R = \frac{\text{number of times that } y_B > y_A}{\text{number of times that } y_A > y_B}$$

which is the proportion of the winning probabilities. As the expected change in player A's wealth in one play is  $x_B P(\text{A wins}) - x_A P(\text{B wins})$ , the game will be fair if this quantity is zero or if

$$x_B = \frac{P(\text{B wins})}{P(\text{A wins})} x_A$$

i.e. the winning probabilities are proportional to the stake values.

Student exercise 2: Simulate the game with the alternative suggestions about the strategy to indicate the winner. To quantify the advantage player B gets, we count the number of times the result is in favor of each of the players. In the cases where the outcome is undecided the toss is repeated. Do the exercise for both cases concerning the stakes as in exercise 1. Discuss your findings with the other students.

The theoretical results about the outcome if  $y_B = \max(y_1, y_2)$  can be represented by means of the probabilities  $P(y_A = i, y_B = j) = \frac{2^{j-1}}{6^j}$  with  $\sum_{i=1}^6 \sum_{j=1}^6 P(y_A = i, y_B = j) = 1$ .

Another representation can be given by the chance that B will win (i.e.  $y_A < y_B$ ) as a function of the outcomes  $y_1$  and  $y_2$  for  $y_B = \max(y_1, y_2)$  as it is presented in Table 1. Table 2 shows the partitioning of the frequencies according to the winner. For example: when  $y_A = 2$ , player A will win in case of the unique combination  $(y_1 = 1, y_2 = 1)$ . The game will be undecided if  $(y_1 = 1, y_2 = 2)$ ,  $(y_1 = 2, y_2 = 1)$  and  $(y_1 = 2, y_2 = 2)$ . For all other combinations  $(y_1, y_2)$  B will be the winner.

Student exercise 3: Table 2 shows that after one play the chance that A is indicated as the winner is  $\frac{55}{216}$ . But with ties (undecided cases) the play is made over again. Here A can turn out to be the winner, but again this play can be undecided, etc. Can you calculate  $P(\text{A wins})$  taking ties into account to consider the effect of ties.

|       | $y_2$ | 1   | 2   | 3   | 4   | 5   | 6   |
|-------|-------|-----|-----|-----|-----|-----|-----|
| $y_1$ | 1     | 0   | 1/6 | 2/6 | 3/6 | 4/6 | 5/6 |
| 2     | 1/6   | 1/6 | 2/6 | 3/6 | 4/6 | 5/6 |     |
| 3     | 2/6   | 2/6 | 2/6 | 3/6 | 4/6 | 5/6 |     |
| 4     | 3/6   | 3/6 | 3/6 | 3/6 | 4/6 | 5/6 |     |
| 5     | 4/6   | 4/6 | 4/6 | 4/6 | 4/6 | 5/6 |     |
| 6     | 5/6   | 5/6 | 5/6 | 5/6 | 5/6 | 5/6 |     |

Table 1. Chance that B will win in case of  $y_1 = i$  and  $y_2 = j$  for  $y_B = \max(y_1, y_2)$ .

| result player A | number of results in favour of A | number of results in favour of B | number of undecided results |
|-----------------|----------------------------------|----------------------------------|-----------------------------|
| 1               | 0                                | 35                               | 1                           |
| 2               | 1                                | 32                               | 3                           |
| 3               | 4                                | 27                               | 5                           |
| 4               | 9                                | 20                               | 7                           |
| 5               | 16                               | 11                               | 9                           |
| 6               | 25                               | 0                                | 11                          |
|                 | 55                               | 125                              | 36                          |

Table 2. Counting the winning and undecided outcomes if  $y_B = \max(y_1, y_2)$  for all possible results of the die of A.

Similar analyses can be made for different definitions of  $y_B$ .

Student exercise 4: Make similar representations as in Table 1 or Table 2 for  $y_B = \frac{y_1 + y_2}{2}$  and  $y_B = y_1 + y_2$ .

Student exercise 5: Show that  $P(\text{A wins}) = 0.3058$  in case of  $y_B = \max(y_1, y_2)$ .

The previous counting results show that the game is made fair in the following cases:

A:  $x_B = 2.27 x_A$  and  $y_B = \max(y_1, y_2)$

B:  $x_A = x_B$  and  $y_B = \frac{y_1 + y_2}{2}$

C:  $x_B = 9.05 x_A$  and  $y_B = y_1 + y_2$

### 3. Simulations by students

This case of decision rules based on die tossing contributes to the deeper understanding of probability in combination with gain or loss. It can be used as an alternative way to let students think about probability beyond the classical approach of textbooks [2]. This example is suitable for students who only have the first notions of probability theory. My experience is that learning by doing makes the students curious about the results, stimulates meaningful discussions among students. It also awakens their critical thinking. My students made a simulation with  $n = 15$  of the fair game with the results in Table 6. The simulation could convince them about the importance of large simulations to recognize the theoretical results [1] [3]. By changing  $n$ , they could observe that the variance of the totals as in Table 6 is smaller with larger samples.

| $x_A$  | $x_B$ | $y_A$ | $y_1$ | $y_2$ | $y_B = \max(y_1, y_2)$ | net prize for A | net prize for B |
|--------|-------|-------|-------|-------|------------------------|-----------------|-----------------|
| 1      | 2.27  | 3     | 3     | 5     | 5                      | -1              | +1              |
| 3      | 6.81  | 5     | 2     | 3     | 3                      | +6.81           | -6.81           |
| 1      | 2.27  | 4     | 1     | 5     | 5                      | -1              | +1              |
| 1      | 2.27  | 2     | 4     | 5     | 5                      | -1              | +1              |
| 2      | 4.54  | 5     | 4     | 6     | 6                      | -2              | +2              |
| 1      | 2.27  | 3     | 5     | 4     | 4                      | -1              | +1              |
| 1      | 2.27  | 1     | 3     | 2     | 3                      | -1              | +1              |
| 3      | 6.81  | 5     | 4     | 3     | 4                      | +6.81           | -6.81           |
| 5      | 11.35 | 5     | 2     | 3     | 3                      | +11.35          | -11.35          |
| 3      | 6.81  | 2     | 5     | 3     | 5                      | -3              | +3              |
| 1      | 2.27  | 3     | 1     | 4     | 4                      | -1              | +1              |
| 2      | 4.54  | 4     | 2     | 3     | 3                      | +4.54           | -4.54           |
| 1      | 2.27  | 6     | 4     | 3     | 4                      | +2.27           | -2.27           |
| 4      | 9.08  | 3     | 2     | 4     | 4                      | -4              | 4               |
| 1      | 2.27  | 1     | 1     | 3     | 3                      | -1              | 1               |
| total= |       |       |       |       |                        | 15.78           | -15.78          |

Table 6: Simulation of the fair game in case of  $y_B = \max(y_1, y_2)$

## 4. Extensions

### 4.1. Variable number of outcomes

Throwing a die gives 6 possible outcomes. By using another random generator this number can be generalized to  $n$ . In the following situations the ratio  $R$  is described as a function of  $n$ .

(a)  $y_B = \max(y_1, y_2)$

If A has  $j$ , A will be the winner when  $y_1$  and  $y_2$  are both smaller than  $j$ . This can be accomplished  $(j-1)^2$  ways. Summing on the possible  $j$  values for player A, makes that the number of times the result is in favor of player A can be written as

$$f_A(n) = \sum_{j=1}^n (j-1)^2 = \sum_{j=1}^{n-1} j^2 = \frac{1}{6} n(n-1)(2n-1).$$

If A has  $j$ , we will have a tie if  $(y_1 = j \text{ and } y_2 \leq j)$  or  $(y_2 = j \text{ and } y_1 \leq j)$ . This can be accomplished  $(2j-1)$  ways. This brings the total number of undecided cases to  $n^2$ .

The total number of times the result is in favor of player B can be written as

$$f_B(n) = n^3 - n^2 - \frac{1}{6} n(n-1)(2n-1) = (n-1) \left( \frac{2}{3} n^2 + \frac{1}{6} n \right).$$

This makes the ratio  $R$  equal to

$$\frac{f_B(n)}{f_A(n)} = \frac{4n^2 + n}{2n^2 - n}$$

which tends to 2 when  $n \rightarrow +\infty$ . This is confirmed by Table 7 where for a sample of  $n$  values the amplifying ratio  $R = P(\text{player B wins})/P(\text{player A wins})$ , is given.

|   |   |      |      |      |      |      |
|---|---|------|------|------|------|------|
| n | 2 | 6    | 10   | 20   | 50   | 100  |
| R | 3 | 2.27 | 2.16 | 2.08 | 2.03 | 2.02 |

Table 7: ratio  $R$  if  $y_B = \max(y_1, y_2)$

(b)  $y_B = y_1 + y_2$

If A has  $j$  and  $y_1$  takes the value  $i$ , A will be the winner when  $y_2$  varies from 1 to  $(j-i-1)$  to make  $y_1 + y_2$  smaller than  $j$ . This assumes that  $i$  does not exceed  $(j-2)$ . Summing on the possible  $j$  and  $i$  values, makes that  $f_A(n)$ , the number of times the result is in favor of player A, can be written as

$$\sum_{j=3}^n \sum_{i=1}^{j-2} (j-i-1) = \sum_{j=3}^n \frac{1}{2} (j-1)(j-2) = \sum_{j=1}^{n-2} \frac{1}{2} (j+1)j = \frac{1}{6} n(n-1)(n-2).$$

If A has  $j$ , we will have a tie if  $(y_1 = i \text{ and } y_2 = j-i)$ . Here  $i$  can vary from 1 to  $(j-1)$ . This brings the total number of undecided cases to

$$\sum_{j=2}^n (j-1) = \sum_{j=1}^{n-1} j = \frac{1}{2} n(n-1).$$

The total number of times the result is in favor of player B and the ratio  $R$  can be written as

$$f_B(n) = n^3 - \frac{1}{2} n(n-1) - \frac{1}{6} n(n-1)(n-2) = \frac{1}{6} n(5n^2 + 1).$$

$$\frac{f_B(n)}{f_A(n)} = \frac{5n^2 + 1}{(n-1)(n-2)}$$

The latter tends to 5 when  $n \rightarrow +\infty$ , which is also suggested by Table 8.

|   |    |      |      |      |      |      |
|---|----|------|------|------|------|------|
| n | 3  | 6    | 10   | 20   | 50   | 100  |
| R | 23 | 9.05 | 6.96 | 5.85 | 5.32 | 5.15 |

Table 8: ratio  $R$  if  $y_B = y_1 + y_2$

#### 4.2. Variable number of throws

Another way to generalize the game is when the result of B is the maximum value of  $m$  throws. The sum of the number of times the result is in favor of player A can be written as

$$f_A(n, m) = \sum_{i=1}^{n-1} i^m = \frac{n^{m+1}}{m+1} + \sum_{k=1}^m \frac{B_k}{m-k+1} \binom{m}{k} n^{m-k+1}$$

with  $B_k$  the  $k$ -th Bernoulli number [4]. The highest order term is  $\frac{n^{m+1}}{m+1}$  and will determine the asymptotical behavior. The total number of undecided cases is  $n^m$ . This can be explained as follows: If A has  $j$  then B must also have at least one  $j$  and all other values no larger than  $j$ . Suppose B has  $i$   $j$ 's, this can be accomplished  $\binom{m}{i}$  ways, in the remaining  $(m-i)$  throws values 1 through  $(j-1)$  are possible, this can be accomplished  $(j-1)^{m-i}$  ways. As  $i$  may range from 1 to  $m$ , we have (using the binomial theorem) the count

$$\sum_{i=1}^m \binom{m}{i} (j-1)^{m-i} = [(j-1) + 1]^m - \binom{m}{0} (j-1)^m = j^m - (j-1)^m.$$

Finally, summing on the possible  $j$  values for player A, we have that the total number of undecided cases is

$$\sum_{j=1}^n j^m - (j-1)^m = n^m.$$

The total number of times the result is in favor of player B can be written as

$$f_B(n, m) = n^{m+1} - n^m - f_A(n, m).$$

This makes the ratio  $R$  equal to

$$R = \frac{n^{m+1} - n^m - \frac{n^{m+1}}{m+1} + L}{\frac{n^{m+1}}{m+1} + L},$$

which tends to  $m$  when  $n \rightarrow +\infty$ , which is the generalization of the result obtained for  $m = 2$  in the previous paragraph. Here  $L$  are lower-order terms.

#### 5. Conclusions

Guided by exercises, an overview is given of the discovery of several strategies to determine the winner of the described game. The impact on the winning probability can be the subject of an animated discussion among students on probability. The ratio of the winning probabilities of both players can be used to design a rule leading to a fair game. Only if the stakes have the same proportion as the winning probabilities, the game is fair. Otherwise one of the players will benefit an advantage.

#### References

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