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Laplace Transform and Applications to Electric Circuits

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Abstract

Observing the student data success in math courses of Faculty of Engineering, University of Rijeka, through a series of academic years, it can be concluded that students achieve the best results in the topics within the field of their profession. The reason of this lies in their motivation, but also in a good preparation of their lecturers who try to bring mathematical topic closer to them through a series of practical problems from the domain of their interest. Although the students have reached a sufficient level of mathematical literacy at college, that they can be offered a wider and more complex content, the fact is that mathematics professors must constantly develop, improve, and refine their mathematical knowledge with topics that will match the interests of their students. Only by doing this will they achieve the best results and the highest examination pass rates, and thus the satisfaction of the students. A good example of such a mathematical domain is precisely the Laplace transform, which is a very useful mathematical tool for solving ordinary differential equations. At the same time, Laplace transforms represent an exceptional tool for modeling and solving various problems in the field of electrical circuits for students of electrical engineering. Ordinary differential equations, systems of ordinary differential equations or integro-differential equations are simply translated (transformed) through their use into ordinary algebraic equations or systems of algebraic equations, which are considerably easier to manipulate. Moreover, some more complex problems can be resolved by means of Laplace transforms, which cannot be solved with ordinary differential equations. Laplace transforms will be presented in this work on certain examples, with an interesting use on electric circuits in the way that it is done at our institution.

1. Introduction

In the first academic year, students at the Faculty of Engineering, University of Rijeka, participate in two basic mathematical courses just as the students of all undergraduate programs, Naval Architecture, Electrical Engineering, Computer Science, and Mechanical Engineering. Some of the goals of these courses are to gain basic knowledge and skills in the field of differential and integral calculus and ordinary differential equations (ODEs). Once the students have overcome basic mathematical skills and gained a basic level of knowledge, they deepen and refine their mathematical knowledge on these subjects in senior years. Students of different programs take part in various mathematical lectures that cover topics from their narrow area of interest.

In the second academic year, students of the undergraduate program of Electrical Engineering take the course Mathematics for Engineers. Some of the expected learning outcomes of this course are defining and correctly interpreting the fundamental concepts of the Laplace transform, expressing their basic properties, calculating the Laplace transform of some functions, and determining solutions to some differential equations using the Laplace transform [1]. Although this course is more demanding than the aforementioned one, on average, students achieve better results. The reason for this lies in the greater motivation of students, but also in the good preparation of their professors. An important task of professors is to bring closer material to students through interesting examples, but also mathematics as a science that can be applied in various fields. Generally, students prefer to learn when they see and understand the application of learned lessons in real life, especially in their profession.

The Laplace transform and its application in solving ODEs is a topic that can be explained to the students of Electrical Engineering using the examples in their profession. In this paper, we will show the application of the Laplace transform on electric circuits, as we do it at our Faculty.

2. Mathematical model of electric circuit

The introduction of each new mathematical topic, including the Laplace transform, should be motivated with an example. Our experience has shown that this is most successfully performed by setting up a mathematical model of an electric circuit. This example is particularly well-received by the students because it shows how mathematical tools can be applied in their profession. Undergraduate

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university students of electrical engineering know that the *RLC* circuit is an electrical circuit consisting of three basic components, namely, resistors, inductors, and capacitors (see Fig.1).

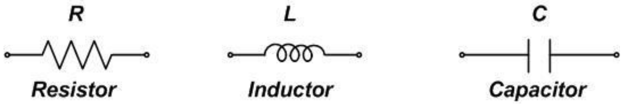

Figure 1: Three basic components in electric circuits.

[4] In it a resistor of resistance *R* Ω (ohms), an inductor of inductance *L* H (henrys), and a capacitor of capacitance *C* F (farads) are wired in a series and connected to an electromotive force *E*(*t*) V (volts) (see Fig.2).

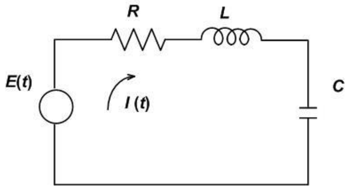

Figure 2: *RLC* circuit.

One wants to find the current *I*(*t*) A(amperes) flowing in the *RLC* circuit for *R*, *L*, *C*, and *E*(*t*), *I*(0), and *Q*(0) given. [4] An ordinary differential equation (ODE) for the current *I*(*t*) in the *RLC* circuit is obtained from the 2^{nd} Kirchhoff's Voltage Law, i.e. the voltage (electromotive force) impressed on a closed loop is equal to the sum of the voltage drops across other elements of the loop. These voltage drops are *RI* for a resistor, *LI* '=*LdI/dt* for an inductor, and *Q/C* for a capacitor, where *Q* coulombs is the charge on the capacitor, related to the current by *I*(*t*)=d*Q*/*dt,* equivalently *Q*(*t*)= ∫*I*(*t*) *dt*. The sum of the voltage drops leads to the equation for the current

$$LI'(t) + RI(t) + \frac{1}{C}Q(t) = E(t),$$

i.e. the integro-differential equation

$$LI'(t) + RI(t) + \frac{1}{C}\int I(t)dt = E(t),$$

as a model for *RLC* circuit with electromotive force *E*(*t*). After differentiating with respect to *t,* we obtain the second order linear ODE with constant coefficients

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t).$$

Students learn how to solve this equation in the first academic year, so they are familiar with the fact that the procedure of finding solution depends on coefficients of the associated homogeneous differential equation as well as on the right-sided function or an electromotive force *E*(*t*), and that it can be very complex.

The question that motivates the students to learn the Laplace transform is whether this problem can be solved by passing differential equations. Solving every mathematical problem is at the same time revealing and it creates something new. The main goal of teaching mathematics at all levels of education, mostly at the academic level, should be to develop creative thinking of every student. In the teaching process, the professor must help the students to discover new mathematical tools and to realize in which circumstances it is more appropriate to use a particular mathematical method for solving them.

After motivating the students with an opening example, it is necessary to introduce general concepts and properties of the Laplace transform.

3. Laplace transform

The Laplace transform (see Fig.3) is an example of an integral transform method. The purpose of using this method is to create a new frequency domain where inputs and outputs are functions of a complex frequency in which it is easier to resolve the placed problem. Results obtained in the new domain can be inverse-transformed to give the results in the original time domain where inputs and outputs are functions of time [3].

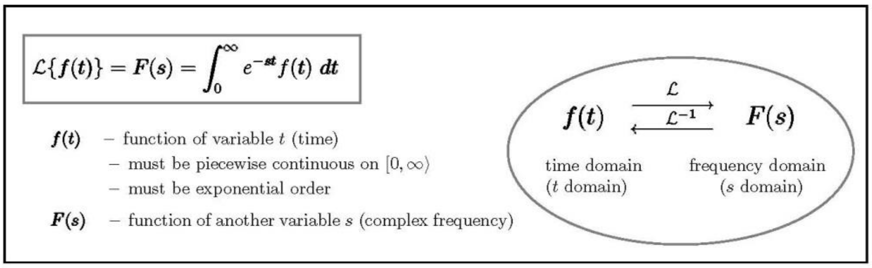

Figure 3: Laplace transform.

First we define the Laplace transform of a function *f*(*t*) (see Fig.3) and then the students compute the transformations of some simple functions directly by using the definition. The properties of the Laplace transform, such as linearity, the first shift property, and derivative of the transform, enable us to find further transformations of some more complex functions without having to compute them directly. The Laplace transform pairs *f*(*t*) and *F*(*s*) of different functions are listed in the transformation table, which students use very well after they have solved all examples from the textbook. The same table is used to find inverse transform called the inverse Laplace transform. The symbol $\Lambda^{-1}\{F(s)\}$ denotes a causal function *f*(*t*), whose Laplace transform is *F*(*s*) [3]. Frequently it is not possible to write down the inverse transform directly by using the table of transforms, so first we need to do some algebraic manipulations on the function *F*(*s*). Usually, students have to determine the inverse Laplace transformation of rational functions. First they have to resolve the function into partial fractions and which they have learned in their first year of studies. Once they have reached the basic knowledge and skills in Laplace transformations, students can then apply them to solving differential equations.

4. Solving linear ODE using Laplace transforms

The Laplace transform method is a powerful tool for solving linear ODEs with constant coefficients. It transforms differential equations into algebraic equations (see Fig.4) and instead of solving differential equations in the *t* domain we need to solve algebraic equations in the *s* domain (see Fig.3). The desired solution of differential equation may be obtained by taking the inverse transform on the solution of an algebraic equation. This method is used also to solve integro-differential equations and ODEs systems.

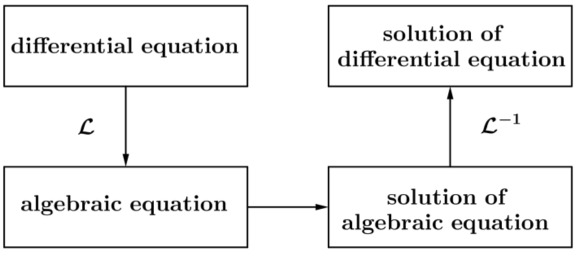

Figure 4:An idea of the Laplace transform.

Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE. The initial conditions are automatically incorporated into the solution and complicated inputs (right side of linear ODEs) can be handled very efficiently [4]. The professor needs to solve at least one ODE using both methods, the classical approach and the Laplace transform, so that the students could understand the advantages of using the Laplace transform method for solving ODEs. This method is particularly powerful for problems with forcing functions that have discontinuities or that represent short impulses or complicated periodic functions. The linear differential equation, in which the forcing function is piecewise-continuous, can be solved in two ways. The first approach is to solve it separately for each of the continuous components (all the derivatives, except the highest, must remain continuous). The second one is to use the step function to specify the forcing function. In engineering applications, we frequently encounter functions whose values change suddenly (switching a voltage on or off in an electrical circuit). Mathematically we describe the switching process by the function called the Unit Step Function (Heaviside function). This function may be used to write a formulation of piecewise-continuous functions. Dirac's delta function models problems which represents phenomena of an impulse nature, i.e. the action of forces of voltages over a short period of time. Such problems can be also solved by using the Laplace transform.

5. Application to the electric circuit

Except for the opening motivational example, in teaching we also use examples for practicing the already learned. [4] On selected examples we will explain what students have learned from Laplace transforms and how they can use it in the field of their profession.

Example 1. Using the Laplace transform, find the current *i*(*t*) in an *RL* circuit with *R*=1Ω, *L*=1H, and *E*(*t*)=*u*(*t*) (see Fig.5), assuming zero initial current, i.e. $i(0) = 0$.

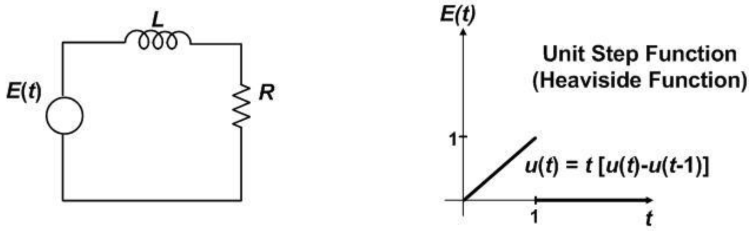

Figure 5: *RL* circuit in Example 1.

The electromotive force *E*(*t*) can be represented by unit step function *t*[*u*(*t*)-*u*(*t*-1)]. Hence, the model for the current *I*(*t*)is the differential equation

$$i'(t) + i(t) = t[u(t) - u(t-1)], \qquad i(0) = 0.$$

Using the Laplace transform,one gets the subsidiary equation

$$sI(s) + I(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}.$$

Solving algebraically for *I*(*s*), simplification and partial fraction expansion gives

$$I(s) = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} - \frac{1}{s^2}e^{-s}.$$

Hence, using the inverse Laplace transform one gets the current

$$i(t) = (t - 1 + e^{t})u(t) - (t-1)u(t-1).$$

Example 2. Using the Laplace transform, find the currents $i_1(t)$ and $i_2(t)$ in the network in Fig.6. with *R*=10Ω, *L*=20H, C=0.05F, *E*(*t*)=20V, and $i_1(0)$ =0, $i_2(0)$=2A.

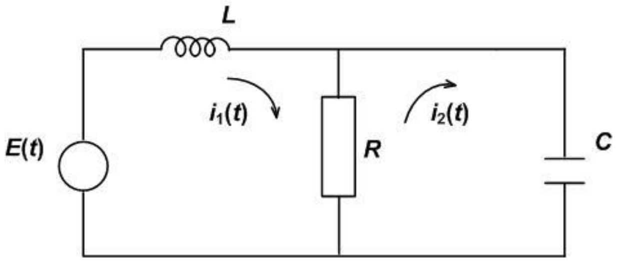

Figure 6: Electrical network in Example 2.

The model of the network is obtained from Kirchhoff's voltage law. For the left circuit one obtains the differential equation

$$20i_1'(t) + 10[i_1(t) - i_2(t)] = 20,$$

and for the right, after differentiation,

$$10[i_2'(t) - i_1'(t)] + 20i_2(t) = 0$$

Using the Laplace transform of a given system of differential equation, one obtains the subsidiary system

$$20sI_1(s) + 10[I_1(s) - I_2(s)] = \frac{20}{s}$$
$$10[sI_2(s) - 2 - sI_1(s)] + 20I_2(s) = 0.$$

Solving algebraically for $I_1(s)$ and $I_2(s)$ gives

$$I_1(s) = 2\left(\frac{1}{s} - \frac{1}{s+1}\right), \quad I_2(s) = \frac{2}{s+1}.$$

The inverse Laplace transform of this gives the solution

$$i_1(t) = 2(1 - e^{-t})u(t), \quad i_1(t) = 2e^{-t}u(t).$$

6. Conclusion

The main purpose of the Laplace transform is the solution of ODEs, integro-differential equations and systems of ODEs. The purpose of using this integral transform method is to create a new domain in which it is easier to resolve the placed problem. Using this method, initial conditions are automatically incorporated into the solution, therefore it is a very powerful tool for solving initial value problems such as those occurring in the investigation of electrical circuits. In this work we presented two examples of showing this application in *RL* electric circuit and electrical network. Those examples are very interesting and useful for students at our Faculty because *RLC* electric circuits are the basic building blocks of large networks in computers and elsewhere.

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