Parameterization of Whirling Inner Circles

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Abstract

Optical illusions attract young people because they are surprising and mysterious. They can also be used as a starting point for a mathematical topic such as parametrization of planar curves. From a mathematical point of view, this paper describes the path of fixed points on a circle while whirling inside a larger circle. A surprising linear path is evoked under specific conditions and is analyzed by mathematical parameterization, which can be used to start discussing parameterizations of planar curves in a calculus course at university first degree level. Moreover, we provide Maple-code to foster student exploration of paths on whirling inner circles. A description of the implementation and students' reactions in our own calculus classes is added.

Keywords: *parameter, modelling, geometry.*

Introduction

Popular internet movies (www.youtube.com/watch?v=pNe6fsaCVtI) show the surprising optical result of dots moving on a straight line when a circle through the origin rotates with fixed point (0, 0) inside the outer circle with center (0,0) and radius *R*. The American University in Washington DC (Rose-Henig & Shapiro, 2013) developed an applet, which visualizes this movement. The amazement awakened by this applet is an exquisite opportunity for mathematics teachers of calculus courses to let students understand this phenomenon by means of geometry and parameterizations (Adams & Essex, 2013) (Harrison & Afima, 1993) of planar curves. A simulating tool can serve as a confirmation of the constructed mathematical model. Controlling the movements of the circles by means of this simulation tool (made with Maple-code (www.maplesoft.com)), will strengthen their deeper understanding of the parameterization.

1.   Parameterization

We want to define a fixed point on an inner circle with radius $\frac{R}{2}$, which is rotating around its center. Its center is moving on a circle with center (0, 0) and radius $\frac{R}{2}$. With names defined in Figure 1, we can write down a vectorial identity

$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = \left\{\frac{R}{2}\cos\varphi, \frac{R}{2}\sin\varphi\right\} + \left\{\frac{R}{2}\cos\alpha, \frac{R}{2}\sin\alpha\right\},$$

so an arbitrary point *P* on the inner circle through the origin can be described as

$$PARA : \begin{cases} x = \dfrac{R}{2}\cos\varphi + \dfrac{R}{2}\cos\alpha \\ y = \dfrac{R}{2}\sin\varphi + \dfrac{R}{2}\sin\alpha \end{cases} \qquad 0 \le \alpha, \varphi \le 2\pi.$$
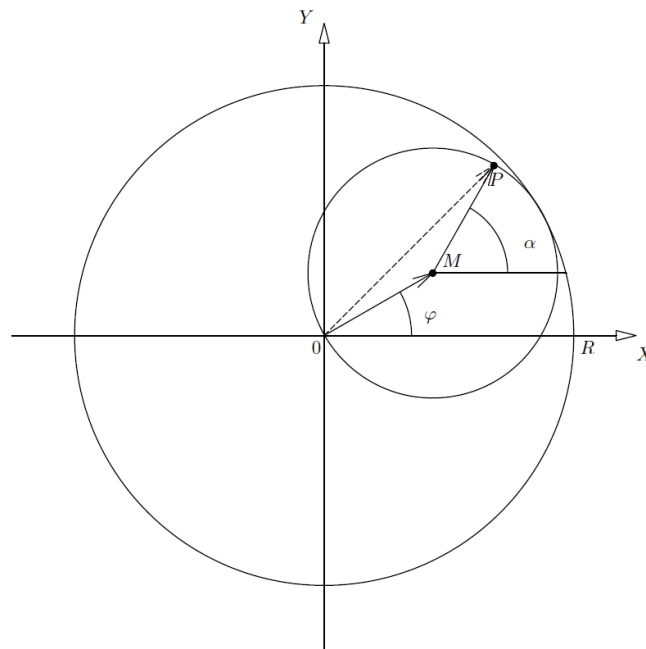
Fig. 1. Notations in the parameterization of the whirling inner circle.

2. Varying parameters

Changing the angle φ makes the inner circle move along the outer circle, while changing the angle α moves P along the inner circle while the center and radius of the latter are fixed. Figure 2 (left) shows the movement caused by varying φ. We start from an initial circle with center $(R_2, 0)$ and radius R_2, which contains a chosen point P. Consecutive circles ($\varphi = i\frac{\pi}{3}, 0 \leq i \leq 5$) with center M_φ arise where only the center M_φ is making a circular movement, but where no movement along the inner circle is present as α is fixed. The dashed line shows the circular path of point P for this variation. When α is the only angle that is varying, a circular movement as in Figure 2 (right) is created along the initial circle with center $(R_2, 0)$ and radius R_2.
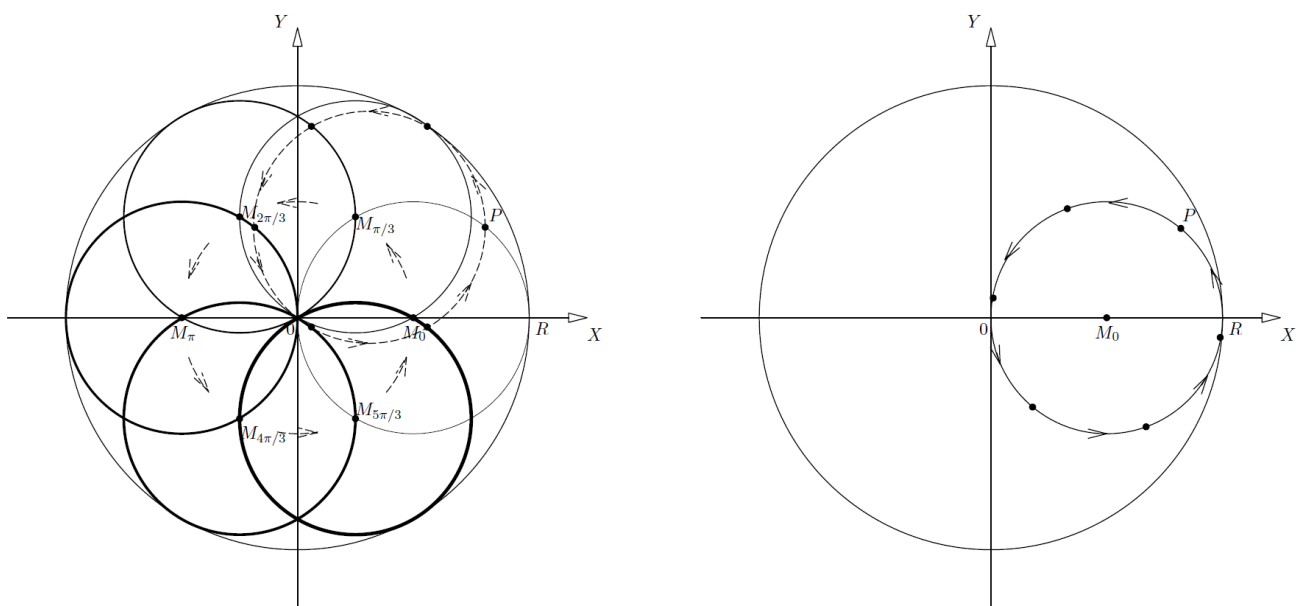

Fig. 2. Circular path described by a point P with varying i (left) and with varying α (right).

When the variations of φ and α are combined, the inner circle is whirling counter clockwise, while meanwhile we move along the inner circle. The movement from point $A_1(x_1, y_1)$ to point $A_2(x_2, y_2)$

parameterized as in *PARA,* will follow a straight path through the origin as in Figure 3 if $\Delta\varphi = -\Delta\alpha$ with $\Delta\varphi = \varphi_2 - \varphi_1$ and $\Delta\alpha = \alpha_2 - \alpha_1$. This can be verified by calculating the slope of the tangent of the curve

$$\begin{cases} x = \dfrac{R}{2}\cos(\varphi_0 + \Delta\varphi) + \dfrac{R}{2}\cos(\alpha_0 - \Delta\varphi) \\ y = \dfrac{R}{2}\sin(\varphi_0 + \Delta\varphi) + \dfrac{R}{2}\sin(\alpha_0 - \Delta\varphi) \end{cases} \qquad 0 \le \Delta\varphi \le 2\pi$$
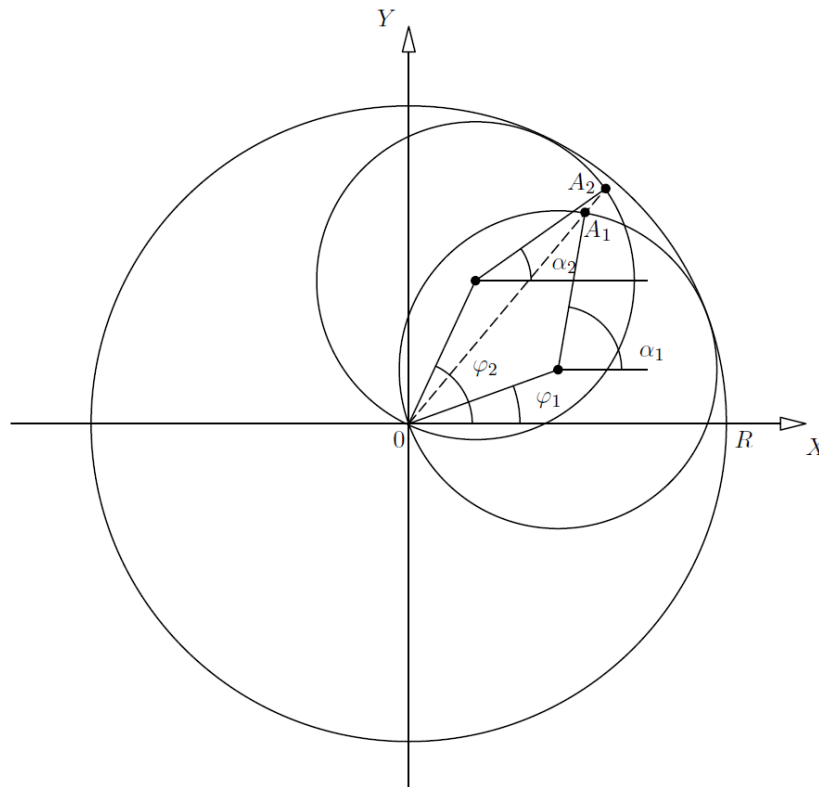

Fig. 3. Linear path of circle points

described by the single parameter $\Delta\varphi$:

$$\frac{dy}{dx} = \frac{dy/d\Delta\varphi}{dx/d\Delta\varphi} = \frac{\cos(\varphi_0 + \Delta\varphi) - \cos(\alpha_0 - \Delta\varphi)}{-\sin(\varphi_0 + \Delta\varphi) + \sin(\alpha_0 - \Delta\varphi)}.$$

Due to the sum-to-product formulas in trigonometry, this expression is equal to y/x (as they are both equal to $\tan\frac{\varphi_0 + \alpha_0}{2}$), the slope of the line through $P(x, y)$ and $O(0, 0)$. This confirms the linear path through the origin.

For the choice $\varphi_0 = 0, \alpha_0 = \frac{5\pi}{18}, \Delta\varphi = \frac{\pi}{6}$, consecutive points

$$P_i\left(\frac{R}{2}(\cos\varphi_i + \cos\alpha_i), \frac{R}{2}(\sin\varphi_i + \sin\alpha_i)\right)$$

with

$$\varphi_i = \varphi_0 + i\,\Delta\varphi$$
$$\alpha_i = \alpha_0 - i\,\Delta\varphi$$

are created for $i \in \{0, 1, 2, 3, 4, 5\}$, which are collinear as φ is increasing at the same speed as α is decreasing. This linear movement of the point P along a line through the origin, is shown in Figure 4.
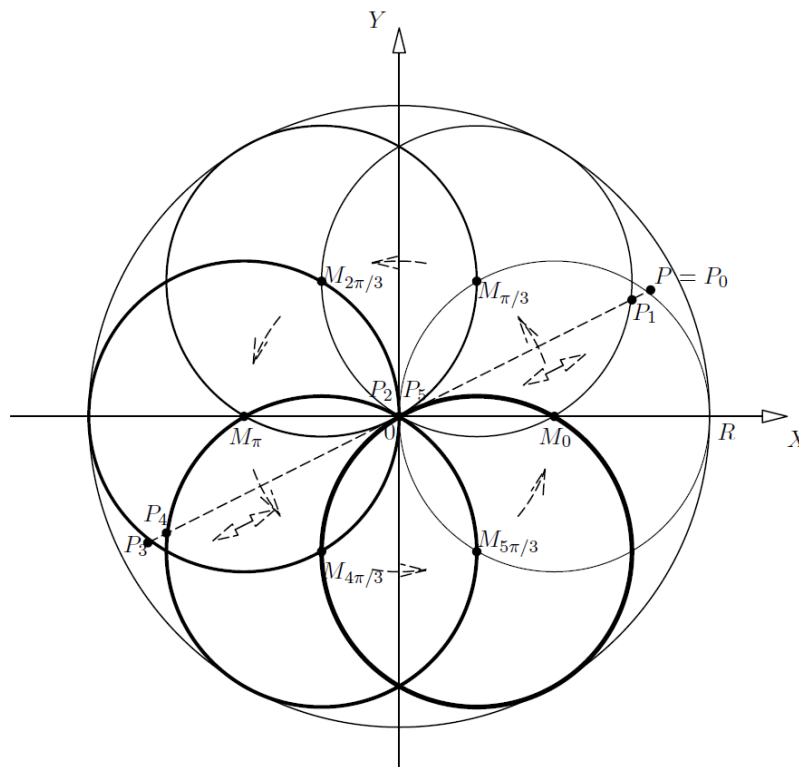
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Fig. 4. Circular path described by a point P while φ is increasing with $\Delta\varphi$ and α is decreasing with $\Delta\varphi$.

3. Simulation tool

For students with some experience in mathematics and coding, an added value can be reached with an additional simulation instruction using Maple-coding. When this approach is beyond the scope of the student group, it can still be used as demonstrative material.

3.1 Creating a linear path when $\Delta\varphi = -\Delta\alpha$

Students can create the path that a point P on the inner circle is describing, by writing the following Maple-code. The graph will convince them of the linear character of the path while the inner circle is rolling inside the outer circle.

```
R:=2:
phi0:=0:
alpha0:=Pi/4:
x := R/2*(cos(phi0+n)+cos(alpha0-n)):
y := R/2*(sin(phi0+n)+sin(alpha0-n)):
verz := {seq([x, y], n = 0 .. 10)}:
with(plots):
pointplot(verz);
```

3.2 Varying one parameter at a time

To experience the influence of each of the determining parameters, the following Maple-code will create an animation which visualizes the variation of φ, α_0 and $\Delta\varphi$ each at a time.

```
phi0:=0:
alpha0:=Pi/4:
x := cos(phi0+n)+cos(alpha0-n);
y := sin(phi0+n)+sin(alpha0-n);
condition := [[phi0=Pi/4,n=0],[alpha0=Pi/8,n=0],
[alpha0=Pi/8,phi0=Pi/4]];
for j to 3 do
co[j,1] := subs(condition[j],x);
```

```
co[j,2] := subs(condition[j],y)
end do;
with(plots);
animatecurve({[co[1,1],co[1,2],alpha0=0..2*Pi],
[co[2,1],co[2,2],phi0=0..2*Pi],
[co[3,1],co[3,2],n=0..2*Pi]},view=[-3..3,-3..3])
```

4. Incorporation in math classes

We evaluated this developed application with our first year engineering students. In their calculus course in the first bachelor year a chapter is dedicated to curves described by parametric equations, for which this application is appropriate. Because a course on programming is only given after the calculus course, the coding in Maple was not feasible. However, the animation tool based on the Maple-code was successful as demonstration material. This was preceded by a collective view of the YouTube video, which evoked students' attention that was far more than on average. The explanation of the mathematical formulas was tackled after the students got a deeper understanding of the problem and notations. This was reached by a ten minutes during exercise to find the parametrization *PARA*. They were asked to work independently and to make use of Figure 1. Two students addressed the lecturer after the lesson. They expressed that they would appreciate if the Maple-code was uploaded in the electronic learning environment, so they could experiment on their own at home. A request that can only be welcomed. A general survey among the students on their opinion about the content of the lesson, revealed that they were pleased to have classes where mathematics is more than proving formulas.

5. Conclusions

For students with moderate interest in mathematics the topic of a circle whirling inside an outer circle can be an attractive subject to discuss parameterizations of planar curves in a calculus course at university first degree level. The subject can be offered to students with limited mathematical background as demonstrative material, while the creation of a simulation tool is a far more challenging instruction, appropriate for students with more developed mathematical and ICT skills.

References

[1] Adams, R. and Essex, C. (2013). Calculus. Canada: Pearson.
[2] Harrison, M.C. and Afima "Parametric curves: an introduction to curve design", Teaching Mathematics and it Applications 4 (1), 1993, p.167-173.
[3] Rose-Henig, A. and Shapiro, A. "A comparison of hypocycloid perception produced by two different elemental constructions", Journal of Vison 13 (9), 2013, p. 819–819.