Tangibles, Construction of Meaning and Math Problem Solving

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1. The problem behind problem solving

A well-documented difficulty of students in basic and secondary education is solving word problems, i.e., mathematics problems starting out of a written-text enunciation. Since problem solving is a highly appreciated educational competence, this difficulty has received a large deal of attention and research [1], [2]. Based on the hint that this difficulty, at least for basic primary education students may be due mainly to the way children these ages understand arithmetical operations, Alandra academic team designed, developed and validated a systematic and coherent program for basic mathematics learning named *Math in Color*, comprehending 23 work-notebooks and a kit of manipulatives.

The main feature of the use of manipulatives in this program is that it is prescribed on the syllabus contained in the work-notebooks, and it is not left to the teacher to devise how to apply them in the classroom. Evidence suggests that this devising is not a trivial matter for teachers to do and it may well be a major obstacle in an effective use of manipulatives for mathematics teaching and learning [3]. In contrast, in *Math in Color* each student is expected to work by him/herself modeling with tangibles and representing the activities in his/her notebook.

The key design feature of *Math in Color* activities is that it recognizes that mathematics operations are the symbolic representation of an active transformation of quantities by a subject. Thus, activities with manipulatives are not considered a mere illustration or example of a mathematics principle or operation. Rather, these activities are thought of to lead children to experience by themselves the quantity transformations possible under an empirical logic and then represent them in drawings, words and finally symbols. This sort of transformations is at the foundation of all mathematical thinking [10].

At the end of each notebook, there is a part devoted to work with daily life situations presented as word problems (*counting stories*), including suggestions of strategies for solution.

1.1. Experimental testing

The expectation that this use of manipulatives supported students in conferring meaning to arithmetical operations, and that this construction, at its turn, backed the development of the ability to solve problems was experimentally tested over several years at Alandra's tutoring center and, more recently in a pilot financed by the Secretary of Education of the city of Medellin (Colombia). This pilot lasted from March to November, 2011, and involved 28 teachers, 1,050 students of 1-3rd grades of three public schools. At the end of the experience, a random sample of about 40% of students was asked to answer a post-test, which included a questionnaire to assess reading comprehension of problem enunciations. These results were contrasted with those of similar groups of students of public schools who had not experienced *Math in Color*.

The significant results are shown on Table 1. Students were presented with two types of problems: simple and composite. A simple problem could be solved by using one single arithmetical operation, while a composite problem demanded more than one operation.

Table 1 Percentage of success in reading comprehension and problem solving

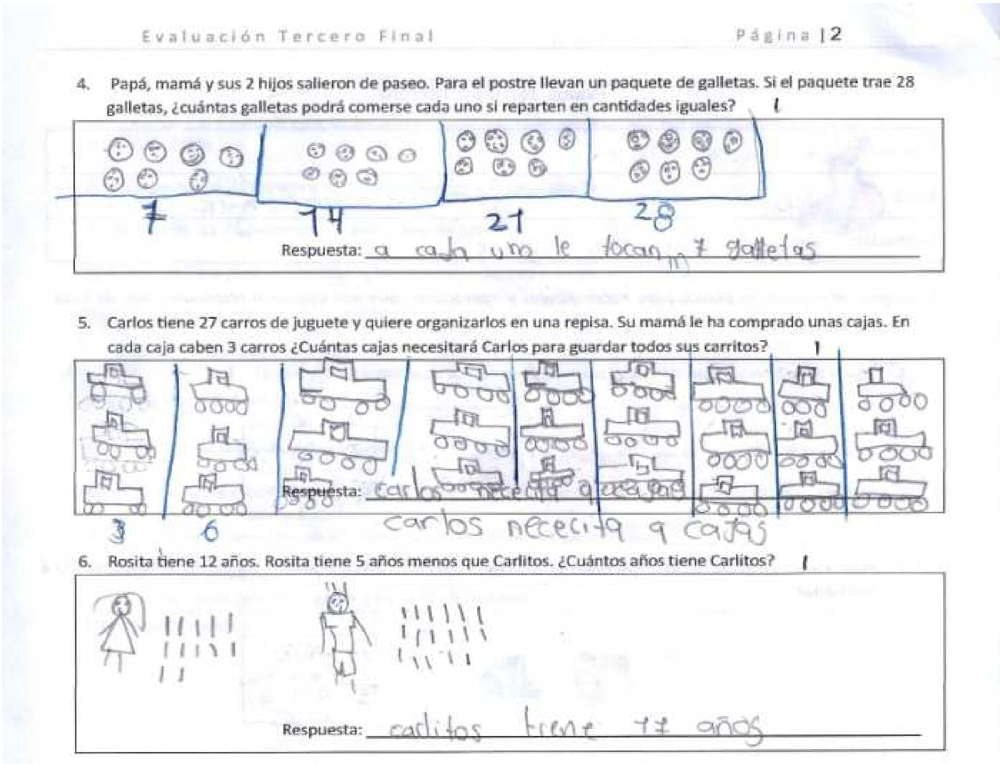
SCHOOL	*Reading comprehension Simple problem*	*Simple problem solving*	*Reading comprehension Composite problem*	*Composite problem solving*
Avg. Experimental groups	78,3%	22,9%	77,5%	29,0%
Avg. Control groups	79,0%	15,4%	70,8%	5,1%

Statistical analysis showed that experimental groups scored higher than the control group and that the differences were significant. Another interesting outcome of these tests was that there were no significant difference as far as reading comprehension was concerned.

Students scoring poorly in reading comprehension could not solve the math problem. However, not all those who scored acceptably or higher could solve the problem. Thus, reading comprehension did not necessarily led to problem solving. What is lacking is the construction of something that could be called mathematical meaning, i.e., constructing a mathematical model out of the meaning of the words in the enunciation: Which is the unknown? What are the relationships among the unknown and the other variables mentioned?

Comparing the outcomes of successful students in both groups with those who scored poorly, an interesting feature arose: Characteristically, students scoring poorly set out to try arithmetical operations with the quantities provided in the problem, whether they made any sense or not (in most cases it did not). In contrast, most of the successful students started out to make drawings attempting to represent the variables and their relationships, as derived from the word problem enunciation (see Figure 1).

Above outcomes suggest that the focus of traditional math education on symbolic algorithms leads to "meaningless" operability, since students appear to understand problem-solving as *applying* some operation. Instead, successful students attempted to make sense out of the enunciation by means of relational drawings. This feature of successful students is strengthened by the activities proposed in *Math in Color*. The focus of the *Math in Color* program is on student´s construction of meaning, both out of everyday-life situations involving quantification/measuring as well as of mathematical symbolism. Algorithms here are not a starting point, but rather the final stage of a process of constructing of meaning. How this could happen is discussed below.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4. Papá, mamá y sus 2 hijos salieron de paseo. Para el postre llevan un paquete de galletas. Si el paquete trae 28 galletas, ¿cuántas galletas podrá comerse cada uno si reparten en cantidades iguales?

7 14 21 28

Respuesta: a cada uno le tocan 7 galletas

5. Carlos tiene 27 carros de juguete y quiere organizarlos en una repisa. Su mamá le ha comprado unas cajas. En cada caja caben 3 carros ¿Cuántas cajas necesitará Carlos para guardar todos sus carritos?

Respuesta: carlos nececita 9 cajas

3 6

carlos nececita 9 cajas

6. Rosita tiene 12 años. Rosita tiene 5 años menos que Carlitos. ¿Cuántos años tiene Carlitos?

Respuesta: carlitos tiene 17 años

Figure 1. This 9 year old boy uses drawings as a graphic strategy to solve all the problems, including those of finding the difference "less than", which usually poses a great difficulty for children these ages.

2. From reading comprehension to building mathematical meaning

The main difficulty confronting students in word problem solving, seems to be related to identifying the mathematical relationships established among the quantities mentioned in the enunciation [4],[5]. ¿How, then, working with tangible manipulatives could support students in establishing mathematical relationships?

The answer proposed by Alandra academic team, which is the foundation for *Math in Color*, is based on the perspective of the student as an active subject transforming quantities of things. This is the most basic level of humans engaged in mathematics, from the natural number in the beginnings of mathematical thinking [6]. Like any other action, transforming quantities of things entails a succession [7] of moments which could be sketched as follows:

Initial quantity $\Rightarrow$ this quantity is transformed $\Rightarrow$ Final quantity

Accordingly, what it is intended in the activities proposed in *Math in Color* is that the student experiences these transformations of quantities, while building awareness of the sequence of moments of the transformation [8].

2.1. Organization is previous to transformation

The enunciation of a word problem would make reference to something that is happening or to somebody's action; involved in both is the transforming of quantities of things. But, how an amount is related to the others? How could this relationship be established? The proposal in *Math in Color* is to set a correspondence between moments of the sequence ⟶ quantity, according to the timeline depicted in the story told in the enunciation.

Here, the tangible manipulatives are organized by the student in a spatial sequence. In those cases in which the quantities correspond to things of the same cardinal set, the possible transformations would be adding or subtracting quantities. For these cases, *Math in Color* proposes activities for organizing manipulatives in space; here, a position corresponds to one moment of the sequence, as shown in the following example in Figure 2.

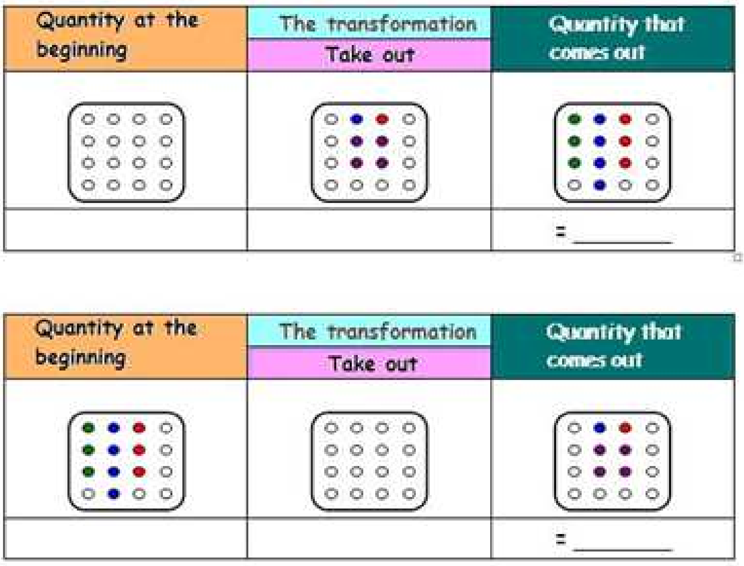

Figure 2. Sample of syllabus proposing activities for adding/subtracting transformations. The manipulatives are rubber foam stencils and plastic balls, known as Alandra's boxes

The student can visualize the moments of the transformation as a spatial sequence. This type of activities, which could be varied to include even the kind of transformation as an unknown, are devised to support students to be aware of the fact that the unknown quantity in the sequence, whatever it is, the initial, the final, or the amount, should satisfy the conservation of the larger quantity in a part–whole relationship along the moments of the transformation.

For those cases in which the situation proposes adding several times the same amount or distributing a given amount into equal parts, *Math in Color* activities are designed for students to differentiate and make sense of the three quantities involved: a) the resulting quantity; b) the number of things in each group; and c) how *many times* groups are taken. This latter has proven to be troublesome for students of theses ages, and it is so because it is determined by a different set of the cardinal set of things being counted [9].

The activities posed in *Math in Color* for these cases are intended for the children to understand the one–to-several correspondence, or a group of things taken several times, with tangibles (Alandra's boxes, and wooden cubes and rubber foam coins). This relationship between two quantities would lead the student to experience an iterative addition. Inversely, the student may experience distributing an amount by subtraction into several groups.

Experiencing mathematical relationships in this manner, starting from correspondences spatially established with manipulatives, followed by making drawings representing those correspondences, seems to support the construction of meaning for the consequent symbolic expressions. Drawing the correspondences mentioned in the problem enunciation makes it easier for the student to identify the set that will be multiplied, i.e., the number of things in each group, and the set indicating how many times. The student should be able to understand that the set indicating how many times is a reference, but it does not undergo any transformation. That spatial organization is so helpful that

students who have experienced *Math in Color* tend to use a drawing to interpret the problem in the first place, and only after, the symbolic form.
Finally, it should be noted that this type of classroom activities may be, and in fact they are planned with small quantities, not requiring the student to memorize multiplication tables or rules for performing some algorithm. What matters here is that upon the reading comprehension students may confer meaning to the unknown quantity irrespectively of which one it is, and to the relationships among the amounts given in the enunciation. Development of the ability to calculate using large numbers, still so cherished by many mathematics teachers, is only an accessory trait.

3. Conclusions

As was discussed above, outcomes seem to support the idea that the ability for solving word problems is a matter of construction of meaning. In the perspective of Alandra academic team, there is a process of several *frameworks of meaning* involved: First, the comprehension of the text apparently is a necessary but not sufficient condition; children need to understand what is being said in words, as an initial framework of meaning. Second, by means of experimenting mathematical relationships using manipulatives and then representation in drawings and with symbols, the students were able to construct a “mathematical” framework of meaning. Putting these two together, there arises a third framework of meaning where there is clarity as to what is unknown and what are the relationships among the amounts given in the enunciation. From here to solve the problem is a mere procedural matter.
Finally, this experience suggests that there is a need for more of these prescribed programs, not only in mathematics but also in all other curriculum subjects, particularly those involving systematic and highly formalized knowledge. In fact, the whole idea of construction of meaning is about children being able to understand the cultural heritage they are receiving at school of that formal knowledge mankind has built over the centuries. But for children, that formalization is an arrival point, not a starting one, as it was for those men and women who built our scientific and technological culture. This is the challenge for educational researchers and activity designers.

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