The Frames-of-Meaning Hypothesis. A Model for Mathematics Education

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Abstract

Seven- to ten-year children were able to solve arithmetic problems in dailylife contexts by drawing spatial arrays of correspondences and comparisons, without resorting to symbolic algorithms. These children had previously experienced a learning environment (LE) we designed; in this LE children experience arithmetic operations as actions, organizing quantities of tangibles on a surface with the goal of finding a quantity-unknown. Control-group children who had not such experience, instead, did not attain these solutions and showed difficulties using algorithms. Upon examining the drawings of children who had reached solutions, we consider that such drawings may be representations of those arrays of tangibles they could observe as the outcome of their performing along the experimentations. Key aspect here is the ***way of proceeding*** *to arrange quantities, both on a table and on a paper sheet. In these procedural ways, we have observed patterns in the location of known quantities as either one-to-one or one-to-several correspondences, as well as in the comparison of quantities as quantity-parts and quantity-whole. These actions arranging quantities of tangibles are in close proximity to actions/events from children's dailylife surroundings (f. i. one-to-several correspondence setting table for dinner, a dish that breaks and its pieces are glued together…). Thus, in our LE children follow the orderly sequence in those actions that could be familiar to them, but now experimenting with quantity-numbers. As a result, children seem to become (implicitly) aware that if they arrange quantities* ***in such a way****, it will* ***always*** *lead them to visualize the quantity-unknown on the frame-of-arrangement they just set/drew. Onwards, this successful procedure could then be repeated to* ***achieve such goal*** *(schema of action). The raison-d'etre, the signification of this way of performing for the child would then be the outcome achieved through it, in the way of a means-goal frame-of-meaning. Therefore, when facing a new problem in dailylife contexts, she will be in condition to anticipate (assimilation, inference) that she can achieve such goal by performing an already known action (schema). We will show samples of drawings illustrating both experimentations in the LE, as well as reaching solutions.*

Keywords: *Mathematics education; Active teaching/learning; Learning environments;*

1. The setting

Here, we will examine the drawings by 7-10-year children while solving arithmetic word-problems, in an assessment test after experimenting, in a learning environment (LE), arithmetic operations as actions arranging tangibles. Such experimentation took place along eight months with 1.053 students in a Medellin (Colombia) low-income neighborhood.

Approximately 40% of them did not achieve this kind of solutions; a few attempted, unsuccessfully, symbolic solutions. Over 90% of students from a control-group did not attain solutions of any kind.

Examination of those drawings led us to think that children could have effected representations evoking their experiences within the LE. However, another possibility may be that those pictorial frames reflected their dailylife situations; in this case, we would be in front of spontaneous, i.e., "natural" arithmetic solutions.

There persists the question, though, about those children that did not accomplish solutions as pictorial expressions; this fact poses the likelihood that this is not a spontaneous, natural phenomenon. Therefore, the issue here is ¿how such processes of pictorial representations could have been configured as a totality making sense? [i]

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2. Our hypothesis

We consider such drawings as **pictorial representations** by children who have attained a proficiency in accomplishing a kind of specific action involving quantity-numbers, and that this proficiency allows them to efficiently perform in problem-situations within dailylife contexts.[ii] Notwithstanding, achieving such proficiency could demand continued experiences of this sort.

3. Experimenting two actions as a unique event

From some time in history onwards, arithmetic actions could be considered separately from world-of-life-events and, therefore, were symbolically formalized. At the beginning, though, they were factual actions undifferentiated from dailylife-events and performed with tangibles quantities on a surface [3].In their quotidian surroundings, as well, children experience situations involving manipulation of material quantities, whether continuous (containers with liquids, loaves of bread…) or collections of objects (cookies, pencils…).
Today, symbolic algorithms are used to solve problems in several world-of-life-situations. By and large, pupils learn to perform with the algorithms and later practice applying them. The difficulties for schooling that arise from this approach are sufficiently well known. Nevertheless, we need to address the duality posed by this schism between the algorithms, on one side, and the world-of-life-events on the other.
We shall take a look at this duality by means of the implementation at schools of our LE (Matemáticas-a-Color, see www.alandra.org) as mentioned above; this LE comprehends three stages:

- Stage I: Experimenting with quantity-numbers in dailylife events.
- Stage II: Experimenting with factual arithmetic actions in themselves.
- Stage III: Experiencing problem solving in world-of-life contexts.

3.1 LE Stage I. Two undifferentiated actions: The world-of-life-event and the arithmetic-action-in-itself

Initial experiences in our LE are set in world-of-life-events close to children's day-to-day. Either, they are staged (f.i., the game of elevenses) or narratives to be read. The story presents some quantities expressed with a name/symbol of number, which children may recognize. The child-reader would ascertain that, in the story, there is "something to be done": finding a quantity-unknown.[iii] There is a circle-of-action posed in this **undifferentiated event**, thus:

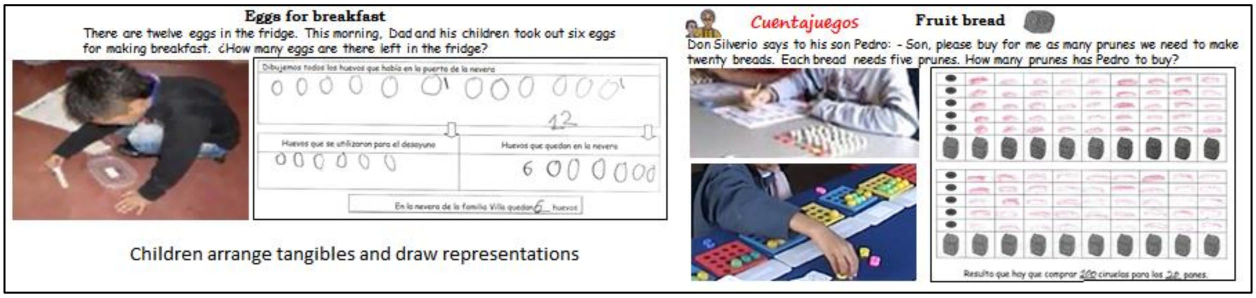
Purpose	→	Means	→	Goal
Some characters must find an unknown quantity.		To find it, they have to "make counts" with known quantities.		The characters find the unknown quantity.

Then, the child places herself in place of one of the characters, to **proceed** to find the unknown quantity. The child-reader herself has to perform the **arithmetic-action-in-itself**, i.e., "making counts". Now, such action is differentiated from the world-of-life-event, even though it depends on it. Respective circle-of-action is:

Purpose	→	Means	→	Goal
The child-reader has to find the unknown quantity.		She will **count** and **arrange** the known quantities.		She will **notice** and **count** the sought-for quantity.

The action the child has to execute is in itself a whole means-goal circle, however dependent on the world-of-life-event. Crucially, the child has to perform her action in **correspondence** with the course of the moments in the world-of-life-event.[iv] Starting from the known quantities, the child is required to set up a frame-of-arrangement with tangibles on a surface. In this frame, each quantity will have a definite role in correspondence with the moments of the world-of-life-event.[v]
Let us see, moment-by-moment, this process in the two following examples (figure next page).

- Initial moment: The **initial-quantity** is placed at this point in the story. Sequentially, the child arranges tangibles (tokens, cubes…), and counts them one-by-one. She already understands that those tangibles "stand for" series of any objects in the world. [vi]
- Transformation moment: It is the point in the story where a **quantity-transformation** is mentioned. The child would effect a change in the initial-quantity by means of the transforming-quantity: She will increase, diminish or share it. The **sense** of the transforming action the child has to perform is expressed in the story, either explicitly or implicitly. In the factual action with tangibles, though, the sense of the action differs largely from operating with algorithms, where there are "jumps-by-rote" (f.i, times-tables). In the factual action, the arithmetic operation becomes a spatial arrangement of tangible quantities.



Children arrange tangibles and draw representations

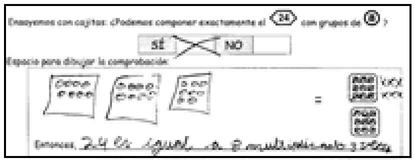
In figure above, example at the left ("Eggs…), the factual action is a setting in one-to-one correspondence between units of the **initial-quantity**, which is a **quantity-whole**, and the **quantity-part** that is "removed" from that quantity-whole. Spatial proximity allows the child to readily "notice-by-comparison" the **quantity-remainder** on her frame-of-arrangement. This is the **quantity-unknown**.

- Final moment: On **her own** arrangement, the child becomes aware of the **quantity-remainder**, and counts it one-by-one with respect to the **quantity-whole**. This "with respect to" is here a spatial proximity. Finally, the child writes down this quantification as the **goal-reached-at** by means of her action, in accordance to the **purpose** in the narrative.

In the example at the right (Fruit bread):

- Initial moment: The **initial-quantity** located here is a **quantity-ones** [vii] (breads). The child serially arranges and counts one-by-one the tangibles.
- Transformation moment: Here, the sense of the transformation is an **allocation by assignment**. [viii] The **quantity-part**, a plurality in itself (group of prunes), is set in one-to-several correspondence as many times as there are **quantity-ones** (breads).In her frame, the child "sees" a series of quantity-parts. However, she still needs to find the **quantity-whole** (total of prunes).
- Final moment: The child counts, one-by-one, a part after the other up to complete the **whole series of quantity-ones**. Thus, she reaches the goal of **her arithmetic-action-in-itself** and writes down her quantification as the **purpose** of the story.

3.2 LE Stage II: Experimenting the arithmetic-action-in-itself



Children repeatedly experience the arithmetic-action-in itself

The action of arranging in correspondence and comparison the **known quantities**, will **always** lead the child to make visible the **unknown quantity** on her frame-of-arrangement. If she proceeds thus, then she will reach the goal of her action. She would have configured a **schema of action** of such procedure.

3.3 LE Stage III: Experiencing word-problem solving in dailylife contexts. The actions reunite again as a unique event

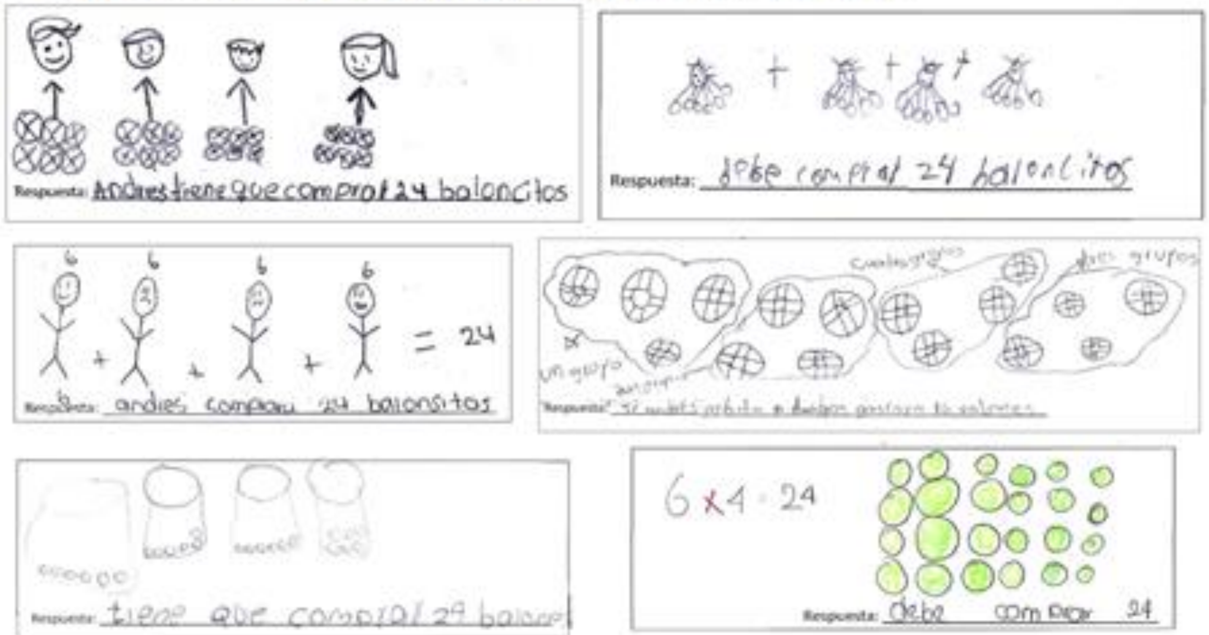
The child confronts problem-statements on an otherwise blank-page. In the story, she "recognizes" (assimilation) the action which has **always** "paid off". In this new situation, then, she will proceed (anticipation, inference) to configure **her frame-of-arrangement** to find the quantity-unknown (accommodation).

Let us notice, in the figures next page, the different pictorial representations. What is **invariant** in them is the setting of quantities in **correspondence** and in spatial proximity, to compare them as **parts of a whole**, to finally bring forward and count the unknown quantity.

Carlos has twenty-seven toy cars and he wants to place them on a shelf. His mother has brought him some boxes. Three cars fit in each box. How many boxes will Carlos need to store all his cars?

Notice the partitions by assignment of the quantity-whole (toy cars), making quantity-parts (groups of cars) in one-to-several correspondence to the quantity-ones (boxes). Children share up to meeting (equal to) the prescribed quantity-whole.

Next Saturday is Andres' birthday and he wants to invite his friends home. To each child, he will give six balls as a souvenir. If Andres invites four friends, how many balls has he to buy?



Notice here the several-to-one correspondence between quantity-parts (groups of balls) and quantity-ones (friends). Children make groups up to meet (equal to) the quantity-ones prescribed in problem statement.

CORRESPONDENCE

MOMENTS IN THE NARRATIVE – MOMENTS IN THE ARITHMETIC-ACTION-IN-ITSELF

	Initial moment	⇨	Transformation moment	⇨	Final moment
Example Cars-Boxes	**Quantity-whole** (Cars) is the *quantity of reference*		**Quantity-part.** Correspondence between cars-boxes *up to* **quantity-whole**		Counting **quantity-parts** to find the total of **quantity-ones** (boxes)
Example Balls-Friends	**Quantity-part** (group of balls)		**Quantity-ones** is the *quantity of reference*. Correspondence between groups of balls-friends *up to* prescribed **quantity-ones**		Counting **quantity-parts** to find **quantity-whole** (total of balls)

Note: Notice that either the **initial quantity** or the **transforming quantity** may be the *quantity of reference*. This will depend upon the sense of the arithmetic-action-in-itself, in correspondence to the moments in the narrative.

4. Conclusions

A factual action of complex structure reveals itself in the above description both of the arithmetic-action-in-itself and its relation to world-of-life-problems. This structure, as well as the diversity of its linkages to world-of-life-events, remains hidden underneath formalized symbolism. The "teaching" modalities relying heavily on procedural rules and/or drawn schematizations (as it is usually done by means of blackboards or tablets) express the procedural ways and schemas of action of the adult who "explains", instead of being the outcome of the **child's own action** as a whole goal-means circle. This fact could explain well the low probability of attainment at school we face today. Arithmetic actions are part of children's world-of-life and they assume their sense nonchalantly. Previous to their involvement with symbolic algorithms, children may experience with tangibles to attain thus an awareness of the arithmetic action as their making under certain requirements, in order to solve problems in dailylife contexts. The raison-d'etre, the signification of this way of performing for the child would then be the outcome achieved through it, in the way of a means-goal frame-of-meaning.

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[i] A previous paper on this subject in [1]

[ii] We have adopted Piaget's approach about the meaning of an action as to what is achieved through it [2].

[iii] A possible way this communicative act could have been historically established in merchandise exchanges, in [4].

[iv] Several authors have examined problem-solving taking into account the moments in the story. However, such examinations omit considering the **child's own actions**, i.e., the child as an **epistemic subject**. F.i., see [5].

[v] In certain cases, children may have to reverse their actions with respect to the moments in the story.

[vi] As empirical representational mediators, a conceptual object [6].

[vii] This quantity, which is readily assumed by children in world-of-life-events, may become problematic in formalizations. See [7].

[viii] For the difference between *partitions **within** a quantity* and *allotting by assignment*, see [4].