Effects of Using Further Math beyond the Curriculum on the Development of High School Student

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Abstract

We, as 11th grade students, conducted a mathematical exploration about the infinite sum of the reciprocals of hexagonal numbers. In our study, we obtained the infinite sum which converged to ln2 by using a further calculus covering differentiation, different methods of integration, Maclaurin series and some tests for convergence of series. What makes our study special is almost all the mathematics we used in our study is beyond our National Curriculum. From the aspect of mathematics education, our claim in this research is the positive effects of studying further mathematics beyond the curriculum at high school level. Students who are following the curriculum are competing with each other in the same track, however, the students who use further mathematics beyond the curriculum have got a head start over the others. Learning mathematics beyond the curriculum which isn't included in the national programs helps students to gain a better vision and intuition over mathematics. Usually, in math education, teachers apply a differentiation in their classes for the competitive students. But this differentiation remains in dealing with challenging problems or a more detailed vision of the same topic. Moreover, further mathematics can be taught by means of flipped classroom activities which is supported by teacher's private tutoring out of class hours. The further calculus we learned during out of class hours covers the first year of university mathematics and we managed to understand and apply the mathematics we learned to another fields of mathematics. That's why we claim, students who are able to understand mathematics beyond the curriculum and who aim to be engineers or mathematicians must learn these topics in high school. This will motivate the students and let them taste the power of mathematics before they start their university career. As G. H. Hardy stated in his 1940 memoir, "No mathematician should ever allow himself to forget that mathematics, more than any other art or science, is a young man's game." [1]

Keywords:

1. Aim

Our aim is to show the convergence of the series obtained by the reciprocals of triangular and hexagonal numbers and to find the value where these series converges.

2. Method

$P_d(n)$: Let n be nth term of a polygonal(figurate) number sequence and d be the number of vertices of a polygon. Then the formula for the nth term is

$$P_d(n) = \frac{(d-2)n^2+(4-d)n}{2} \quad (1)$$

For example, if we substitute d=3, we obtain the formula for triangular numbers as, $P_3(n) = \frac{n(n+1)}{2}$.
Formula for the sum of triangular numbers is:

$$f(n) = \frac{n(n+1)(n+2)}{6}$$

The formula of the sum can be proven easily by mathematical induction.
Sum of the reciprocals of polygonal numbers can be written as follows:

$$\sum_{n=1}^{\infty} \frac{1}{P_d(n)} = \sum_{n=1}^{\infty} \frac{2}{(d-2)n^2+(4-d)n} \quad (2)$$

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In this formula, d >3 and when d=4 general term turns out to be the famous Basel problem; “Sum of the reciprocals of squared numbers”. In our study we will focus on the infinite sum of reciprocals of triangular and hexagonal numbers.

2.1 Convergence of Reciprocals of Triangular Numbers

Let’s substitute 3 for d to find the sum of the triangular numbers (T_∞)

$$T_\infty = \sum_{n=1}^{\infty} \frac{n^2+n}{2}$$

It is obvious that the series diverges considering the nth term divergence test, since the limit of the general term is not equal to zero. Let’s denote the sum of the reciprocals of triangular numbers by S_∞, where

$$S_\infty = \sum_{n=1}^{\infty} \frac{2}{n^2+n} \quad (3)$$

Even though the limit being 0 is a necessary condition for convergence, it is not sufficient. The limit of the general term of this series is 0, it satisfies the necessary condition, however it is not sufficient. So, in order to determine its convergence and also where to converge, we will first decompose the telescoping series into partial fractions and try to find the limit by simplification.

2.2 Convergence of Reciprocals of Hexagonal Numbers

Similarly, in the formula (2), the sum of the reciprocals of the hexagonal numbers can be obtained by substituting d = 6:

$$S_\infty = \sum_{n=1}^{\infty} \frac{2}{4n^2-2n} \quad (4)$$

Firstly, we will apply nth term divergence test.

$$\lim_{n\to\infty} \frac{2}{4n^2-2n} = 0$$

Since the limit is “0”, we cannot decide whether it is convergent or divergent.
Therefore, we used limit comparison test. In this sense we compared our series with $\sum_{n=1}^{\infty} \frac{1}{n^2}$
According p series test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent is p >1. The limit comparison test is defined for two series with general terms which are a_n and b_n where $a_n > 0, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, $L \in \mathcal{R}$. Then either both of the series are convergent or divergent. [2]
If we apply limit comparison test for the series with general terms $\frac{2}{4n^2-2n}$ and $\frac{1}{n^2}$, we obtain:

$$\lim_{n\to\infty} \frac{\frac{2}{4n^2-2n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^2}{4n^2-2n} = \frac{1}{2} \in \mathcal{R}$$

Since limit is a real number, our series is convergent. Once we are sure of this, we will first try to determine the value of the convergence of the infinite sum with the help of technology. We will then find the exact value of the convergence by using appropriate and elegant use of calculus.

2.3 Infinite Sum of Reciprocals of Triangular Numbers

In order to find the sum of infinite terms of the sequence defined by the formula (3), first the general term will be decomposed into its partial fractions as follows.

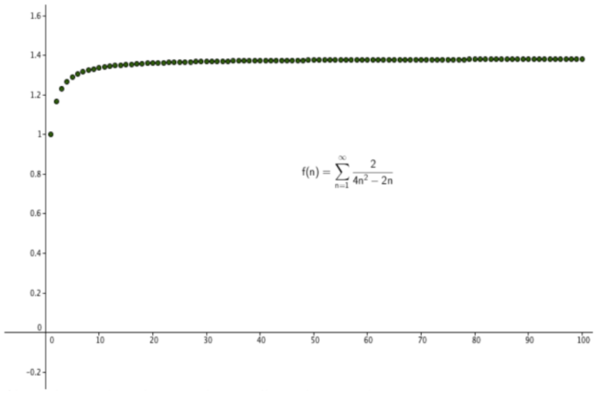
$$\sum_{n=1}^{\infty}\frac{2}{n^2+n}=\sum_{n=1}^{\infty}\left(\frac{2}{n}-\frac{2}{(n+1)}\right)=\lim_{k\to\infty}\sum_{n=1}^{k}\left(\frac{2}{n}-\frac{2}{n+1}\right)$$

$$\lim_{k\to\infty}\left[\left(\frac{2}{1}-\frac{2}{2}\right)+\left(\frac{2}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{2}{4}\right)+\cdots+\left(\frac{2}{k}-\frac{2}{k+1}\right)\right]=\lim_{k\to\infty}\left(2-\frac{2}{k+1}\right)=2$$

Thus, the infinite sum of the reciprocals of triangular numbers converges to “2”.

2.4 Infinite Sum of Reciprocals of Hexagonal Numbers

Firstly, we thought that, the method defined for triangular numbers can also be used in hexagonal numbers since each hexagonal number is a triangular number at the same time. But we could not reach any result from this approach. The limit comparison test we did. shows that the series is convergent, but it does not show to which value it converges. Therefore, we inserted our equation (4) into Ti Nspire calculator in the domain where n∈ [1,100].



Graph 1: The partial sums of first 100 terms for reciprocals of Hexagonal numbers (Geogebra 4.0)

As the graph shows, the sum of the first 100 terms is converging to 1.381306861. However, the main problem here is that the calculator we used has a maximum of 9-digits. This raises the doubt that the value found is still incomplete, that is a very high probability that it is an irrational number. So, for the rest of our project, we will try to find the exact value for convergence.
With the purpose of simplifying the summation for the further operations,

$$S=2\sum_{n=1}^{\infty}\frac{1}{n(4n-2)}$$

Let S = 2A, where $A=\sum_{n=1}^{\infty}\frac{1}{n(4n-2)}$, $A=\sum_{n=1}^{\infty}\frac{1}{n(4n-2)}x^{4n-2}, x=1$

Here we multiplied the general term by x^{4n-2} where x = 1. Our goal here is to be able to expand the infinite sum according to a variable and to obtain a series which resembles to Maclaurin series expansion for $\ln(1-x)$.

$$A(x)=\frac{x^2}{2}+\frac{x^6}{2\times 6}+\frac{x^{10}}{3\times 10}+\frac{x^{14}}{4\times 14}+\cdots+\frac{x^{4n-2}}{n(4n-2)}+\cdots, x=1(5)$$

If we take the derivative of the function A(x) at this stage, the structure we obtain is similar to the Maclaurin series for the function $f(x)=\ln(1-x)$ where its Maclaurin series is:

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n} = -\frac{x}{1}-\frac{x^2}{2}-\frac{x^3}{3}+\ldots$$

$$A'(x) = \frac{x}{1}+\frac{x^5}{2}+\frac{x^9}{3}+\frac{x^{13}}{4}+\cdots, x = 1$$

Accordingly, $A'(x)$can be written as:

$$A'(x) = -\frac{1}{x^3}\ln(1-x^4)$$

Consequently if we write the Maclaurin series for $ln(1-x^4)$:

$$= -\frac{1}{x^3}\left(-x^4-\frac{x^8}{2}-\frac{x^{12}}{3}+\cdots\right)$$

$$A'(x) = \frac{x}{1}+\frac{x^5}{2}+\frac{x^9}{3}\ldots.$$

Therefore, it is obvious that $-\frac{1}{x^3}ln(1-x^4)$ is the anti-derivative of $A(x)$.

$$A(x) = \int -\frac{1}{x^3}ln(1-x^4)\,dx$$

Here we will solve the integral using integration by parts.

$$u = ln(1-x^4) \qquad v = \frac{1}{2x^2}$$

$$du = -\frac{4x^3}{1-x^4}dx \qquad dv = -\frac{1}{x^3}dx$$

$$A(x) = \frac{ln(1-x^4)}{2x^2} - \int \frac{2x}{x^4-1}dx$$

For the integral $\int \frac{2x}{x^4-1}dx$, we will use method of u-substitution, where

$$u = x^2, \frac{du}{dx} = 2x$$

$$\int \frac{2x}{x^4-1}dx = \int \frac{1}{u^2-1}du$$

By decomposing the integral into simple fractions.

$$\int \frac{1}{(u-1)(u+1)}du = -\frac{1}{2}\int \frac{1}{u+1}du + \frac{1}{2}\int \frac{1}{u-1}du$$

$$= \frac{1}{2}(-(ln(u+1) + (ln(u-1)))$$

$$\int -\frac{1}{x^3}ln(1-x^4)\,dx = \frac{1}{2}((-ln(1-x^2) + ln(x^2+1) + \frac{ln(1-x^4)}{x^2}) + C$$

Considering the previous steps, it is obvious that integration constant should be “0”. In the Equation (5), x has been regarded as 1. Therefore, at this stage, we will get the limit of the sum when x approaches to 1^-. Because inner part of the logarithm cannot be negative.

$$A = \lim_{x\to 1^-}\frac{1}{2}\left(-ln(1-x^2) + ln(x^2+1) + \frac{ln(1-x^4)}{x^2}\right) = \infty - \infty$$

In order to find the limit above, first we substituted 1-x^2 = t and then applied L'Hopital rule. Hence, the limit is obtained as *ln2.*

Since A = *ln*2 and S = 2A, then S = 2*ln*2.

$S = 2\,ln(2) \approx 1.386294361$ (Rounded to 9 digits)

This value is almost the same with the one obtained by using technology, giving the exact value of infinite sum.

2. Conclusion

In our exploration, we showed the convergence of the series obtained by the reciprocals of triangular and hexagonal numbers by using different convergence tests. In order to find the value where the series converges, we used a telescopic method for triangular numbers. Since hexagonal numbers is a subset of triangular numbers, we tried to apply a similar method for hexagonal numbers. But we couldn't. Therefore, firstly we used to technology to obtain an approximated value. Then, with the help of a further calculus, which isn't included in our high school curriculum, we obtained the exact value of the approximation as *"ln2"*.

Nowadays the formal mathematical education in high schools serves some certain but limited topics to the students all around the world and it's not personalized for students. In formal education, students progress are determined by their exam grades which labels students as successful or not. Being competent in math isn't about exam grades, any student can achieve much more if they are given the chance for further and deeper studies. For example, famous French mathematician Evariste Galois passed away when he was 21 years old and the contributions that he had done to mathematics are remarkable. However students in our times must wait till 11 th grade to learn logarithm or 12th grade for calculus. Therefore, a student must be 17 or 18 years old to learn these subjects. On the other hand Galois solved a problem that had been proposed 200 years ago when he was only 19 years old. Almost all mathematicians make a splash in their early ages. It can be claimed that there must be students like Galois in today's world, however, their prime time are past while waiting for to be educated. Education systems should provide opportunities for geniuses, hence education system must be revised and personalized.

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