Differentiated Learning with Mathematical Software: Orthogonal Curves from Wire Grid to Differential Equation

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Abstract
In math classes orthogonal trajectories are mostly presented as a purely mathematical issue: curves where at every cross point the tangent lines are perpendicular to each other. Mathematical software however can offer an alternative and more appealing way to introduce orthogonal trajectories: a visual approach guarantees a deeper understanding of the mathematical theory. Moreover mathematical software offers the opportunity to incorporate the topic at different levels: introductory in a more narrative way, intermediary by a programming code to generate the graphical solution or mathematically advanced by a mathematical analysis based on differential equations that can be solved by mathematical software. This makes it attractive for a varied use by the mathematics teacher of a first year calculus course in higher education for students with or without programming experience. Our students, who were part of an experience with this approach, witnessed that computer code and theoretical mathematical analysis can go together and even reinforce each other by confirming the results. We also want to show that this approach to discussing orthogonal curves is suited within interdisciplinary engineering education as it is related to different application fields, e.g. finding the path of the steepest ascent, wire grid representations, ordinary differential equations...

1. Introduction
In literature [7] the importance of visualization tools to improve mathematical understanding, is commonly accepted, but Liang and Sedig [5] stressed the importance of the ability of the tool to cater for the specific needs of the student. With this paper we want to demonstrate the tailor-made use of mathematical software to educate students about orthogonal curves. The notion of orthogonal trajectories/curves is mostly presented as a purely mathematical issue: curves where at every cross point the tangent lines are perpendicular to each other [2][4]. But this interesting topic is a grateful interdisciplinary subject in engineering education as it is related to different application fields, e.g. finding the path of the steepest ascent, wire grid representations, ordinary differential equations... As for some applications fields, more mathematical background is required, it is the mathematics teacher who should offer this subject in the suitable form to his students by means of mathematical software. In the following section, an overview is given of different possibilities to present the orthogonal curves. Comments from our own experience are presented, as we evaluated the approach in our math classes.

2. Incorporation in the math classes
This subject can fit within lessons about geometry as well as lessons about differential equations or computer programming. There are different possibilities for different ages, abilities and technical competences of learners to use it as an attractive example where mathematics can be fun.

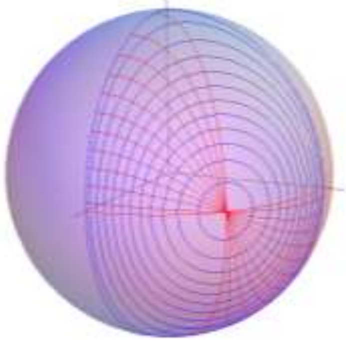

Fig. 1: Parallels and meridians on the globe as orthogonal trajectories

2.1 Lessons at introductory level

A simple example from geography shows that we all already know orthogonal lines: parallels and meridians on a globe are two sets of curves that are orthogonal (see Figure 1). This example can be formalized in a mathematical definition.

Definition: orthogonal trajectories

Two families of curves or trajectories are orthogonal if and only if at each intersection point the tangent lines at both curves respectively are perpendicular.

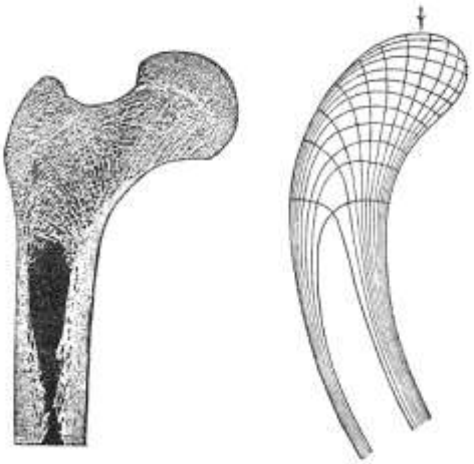

Fig. 2: Spatial representation of objects on a sheet of paper, e.g. a bone

For a more irregular spatial object than a globe, similar orthogonal lines can be drawn. They create a spatial representation of the surface as can be seen for bones from Figure 2. This type of representation is often used in imaging as a wire-type representation. Here students can experiment themselves by outlining on a sheet of paper the orthogonal trajectories of a simple object as a glass for example. More applications of orthogonal trajectories can be found in the engineering world as in electromagnetism the electric field lines and the equipotential lines [8] are orthogonal trajectories too (see Figure 3). An electric field describes the space that surrounds electrically charged particles. Each electrically charged object generates an electric field which permeates the space around it, and exerts pushes or pulls whenever it comes in contact with other charged objects. The electric field generated by a set of charges can be measured by putting a point charge q at a given position. An equipotential is a region in space where every point in it is at the same potential. When the equipotential is zero, there is no push or pull between objects. The electrostatic field created by a positive point charge is pictured as a collection of straight lines which radiate away from the charge. For technical or engineering students this example is especially useful to mention as it must convince them of the omnipresence of mathematics.

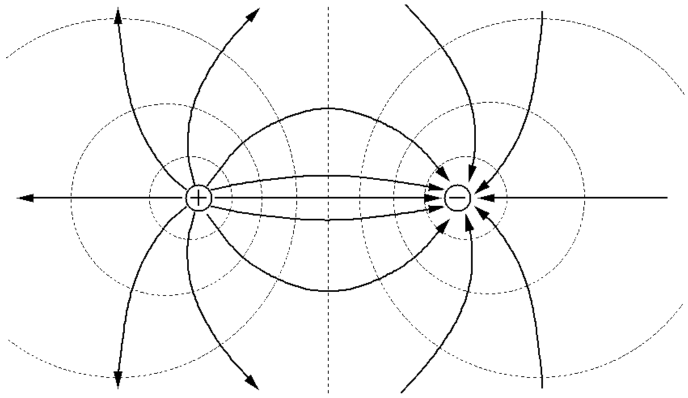

Fig. 3: The electric field and it equipotentials as orthogonal curves

2.2 Lessons at intermediary level

This lesson starts in the same way as that at introductory level, but with much less time actually outlining curves with pencil and pen. Since it can take extensive calculations to generate the differential equation connected to a family of curves and solving the differential equitation for the orthogonal trajectories can also be long job, we make a full analysis using the mathematical tool Maple [1]. With the Maple code, the differential equation associated with the given family of curves (with one constant C) is found as ODE. This is transformed into the differential equation ODEorth of the orthogonal trajectories by substituting $y'(x)$ by $-1/y'(x)$, expressing the orthogonality. ODEorth is solved by means of the dsolve command. This Maple-code [6] is available in Figure 4. Whether the students write the code themselves or not, depends on their acquired skills. By changing $F(x,y)$ they can create varying sets of curves and their corresponding orthogonal curves.

```
restart;
with(DEtools):
with(plots):
F :=(x,y)->y^2-4*c*x;
plotC:=C->implicitplot(eval(F(x,y),c=C),
x=-5..5,y=-5..5):
plot_curves := display( seq( plotC(c),
c=-5..5) ):
diff_F:=diff( F(x,y(x))=0,x):
diff_with_c:=simplify(isolate(diff_F, diff(y(x),x))):
q:=eliminate({F(x,y(x)),diff_with_c},c);
ODE:=q[2][1]=0;
ODEorth:=eval(ODE,diff(y(x),x)=-1/diff(y(x),x));
F_orth:=dsolve(ODEorth,y(x),implicit);
plotCorth:=C->implicitplot(eval(F_orth,[C1=C,y(x)=y]),
x=-5..5,y=-5..5,grid=[50,50]):
plot_curves_orth:=display(seq(plotCorth(c),c=0..10)):
display({plot_curves,plot_curves_orth});
```

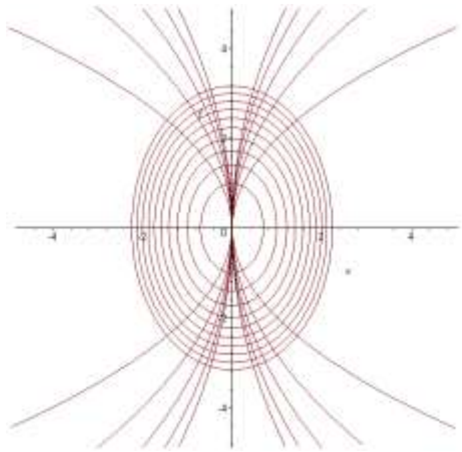

Fig. 4: The curves $y^2 = 4Cx$ and their orthogonal curves as a result of Maple code.

2.3 Lessons at advanced level

a) Solving differential equations

As a family of curves can be interpreted as the solution of an ordinary differential equation (ODE), this topic fits into a lesson about solving first order differential equations.

Example: *Find the orthogonal trajectories of the curves* $x = Ce^{-y^2}$ *(*C *= real constant).*

After differentiating $x = Ce^{-y^2}$ at both sides with respect to x, one gets $1 = -2C\,y\,y'e^{-y^2}$. Eliminating C out of both expressions, we find the first order differential equation $2\,x\,y\,y' + 1 = 0$. The ODE of the orthogonal trajectory (found by replacing $y'(x)$ by $-1/y'(x)$ is $2\,x\,y = y'$ which can simply be solved by separation of variables: $\int 2\,x\,dx = \int \frac{dy}{y} \Rightarrow x^2 + A = \ln|y| \Rightarrow y = \pm B\,e^{x^2}$, where A and $B = e^A$ are real constants. ♣

b) Path of steepest ascent

Recall that the direction of the path of steepest ascent along a surface described by $z = G(x,y)$, is always that of the gradient vector $grad(G) = \nabla G$. Thus, when this path is projected onto the $XY-$plane, it will be orthogonal to each of the level curves $G(x,y) = C$ which it intersects. This makes the path an orthogonal trajectory of the family of curves $G(x,y) = C$.

Example: *Find the path of steepest ascent of the surface* $z=\frac{x^2}{4}-\frac{y^2}{9}$ *at the point where* $x=\frac{1}{4}$ *and* $y=3$.

The work to find the orthogonal curve through $(\frac{1}{4},3,G(\frac{1}{4},3))$ can be done by hand or by using Maple. The following Maple-code generates a 3D-plot of the surface $z=\frac{x^2}{4}-\frac{y^2}{9}$ together with the path of steepest ascent.

```
restart:with(plots):
with(plottools):
G:=(x,y)->x^2/4-y^2/9:
l1:=line([0,0,0],[2.5,0,0]):
l2:=line([0,0,0],[0,3.5,0]):
l3:=line([0,0,0],[0,0,1]):
s1:=textplot3d([2.9,0,-0.3,‘x‘]):
s2:=textplot3d([0,4,-0.3,‘y‘]):
s3:=textplot3d([0,0.45,1.3,‘z‘]):
t1:=polygon([[2.5,-0.3,0],[2.9,0,0],[2.5,0.3,0]],style=patchnogrid):
t2:=polygon([[-0.25,3.5,0],[0,4,0],[0.25,3.5,0]],style=patchnogrid):
t3:=polygon([[0,-0.3,1],[0,0.3,1],[0,0,1.3]],style=patchnogrid):
surface:=plot3d(G(x,y),x=-2..2,y=-sqrt(9-9*x^2/4)..sqrt(9-9*x^2/4),style=wireframe,grid=[50,40]):
curve3d:=spacecurve([t,3/(2^(8/9)*t^(4/9)),G(t,3/(2^(8/9)*t^(4/9)))], t=1/4..1.8,thickness=4):
display(surface,l1,l2,l3,t1,t2,t3,s1,s2,s3,curve3d,orientation=[60,41]);
```

3. Evaluation

We evaluated the usefulness of the handling of the orthogonal trajectories by the mathematics teacher. For our engineering students, receiving a calculus course in the first bachelor year which contains differential equations, we have chosen the lesson plan at high level during a lecture added by a computer practicum. The engineering students in ICT received the extra task to create the Maple code themselves, while the other ones (chemistry, construction, ...) got a basic code which they only had to adapt. The few ICT students with a minor interest in the mathematical background of the problem, were strongly motivated by the programming section.

A general survey among the students on their opinion about the contents of the lesson, revealed that they were pleased to see that

- mathematics is omnipresent,
- Maple brings mathematics alive,
- calculations have another function than just keeping students busy.

1. Strongly disagree
2. Disagree
3. Neither agree nor disagree
4. Agree
5. Strongly agree

Table 1: Scores on a five-level Likert scale

The exercise about the path of steepest ascent was difficult.	3.1
The exercise about the path of steepest ascent was interesting.	3.9
Now I have the feeling that I know what orthogonal trajectories are.	4.1
I like to work with Maple because it can visualize functions.	4.4
I needed some help from the teacher or a student to finish the assignment.	2.4

Table 2: Findings ($\overline{l}$ =mean score on a five-level Likert scale) of the 56 engineering students of the Ghent University

Table 2 shows the mean values $\bar{I}$ of the scores contributed by our 56 engineering students (second year of the bachelor program) at the Ghent University on a five-level Likert scale with possible scores as in Table 1. This table quantifies the positive attitude of our students towards the approach and the activities they were asked to do. Most of the students were able to find the expression for the orthogonal trajectories as described in section 2.3. Displaying the graphs with Maple appeared to be a harder task. The students were enthusiastic about the example in section 2.3 concerning the 3D problem of the path of steepest ascent. Only those with skills in programming were able to reconstruct the Maple-code to generate the saddle surface.

4. Conclusion

It is commonly accepted that the learning of mathematics will be enhanced by appealing problems [3]. With the help of mathematical software, the subject of orthogonal trajectories can be addressed at different levels, which makes it attainable for a wide range of students. It convinces students that computer code and theoretical mathematical analysis can go together and even reinforce each other by confirming the results. Handling the same problem from different angles is enriching and stimulating for the deeper understanding. In this paper a demonstration is given of how mathematical software can assist to teach at an adequate level, depending on the personal needs and skills of each student.

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