# Fourier Analysis - Impacts of Mathematics on Other Educational Sciences

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## Abstract

*Mathematics is an essential tool for solving problems in many different areas, particularly those in engineering and* sciences *such as physics, astronomy, and biology. Knowledge of fundamental mathematical laws enables students to be effective in research, discovery, and understanding of scientific principles in other areas*. *This fact is very important for mathematics teachers and professors at faculties of engineering, civil engineering or faculties of science. Mathematical lectures at these faculties are not only concerned with just pure mathematics with abstract concepts, but also with applied mathematics connected with the students' future profession. However, a particular area of mathematics will be more attractive and understandable if its contents fit into the context of engineering or real scientific world problems. A very interesting mathematical field with a wide application in electronics, acoustics, and communications is the Fourier analysis, also known as spectral analysis. The Fourier analysis is the study of how functions can be decomposed into trigonometric or exponential functions. It was named after the French mathematician and physicist Jean Baptiste Joseph Fourier, who lived during the 18th and 19th centuries. He showed that a continuous periodic function can be expressed as an infinite sum of trigonometric or exponential functions with certain amplitudes, periods, and phases. This result, except for the function in the mathematical sense, has had a significant impact on the understanding of the various areas that include oscillations and waves. Although the theory of Fourier series is complicated, the application of these series is rather simple and can be introduced already in high schools on the phenomenon of sound. Fourier integrals and Fourier transforms extended the ideas and techniques of the Fourier series to an arbitrary function that is not necessarily periodic. Since fast finite Fourier transforms are very useful for data compression on cell phones, disc drives, DVDs, and JPEGs, it can be stated that the Fourier analysis is present in our every day life, maybe without us even being aware of it. A whiff of the Fourier analysis and its fascinating application to other science education will be presented in this paper.*

## 1. Introduction

In mathematics, it is often helpful to approximate a complicated function by using simpler or less complicated ones. This idea lies at the heart of the approximation with Taylor and Fourier series, i.e. an infinite sum of specific terms, a topic that is usually dealt with at faculties of engineering. Taylor series is a tool that expresses complicated smooth functions in terms of simple polynomials, but only when one is interested in the behavior of the function in a small neighborhood of a particular point. The requirement smooth means that the function must be infinitely differentiable there, because the coefficients consist of the derivatives of the function at that specific point. In a certain sense, more universal approximations are those with Fourier series, because they do not dependent upon a specific point, and a function need not be infinitely differentiable at any point or differentiable at all to have a Fourier series, but it does need to be integrable. Fourier series are infinite series that use trigonometric or exponential polynomials to approximate piecewise continuous periodic functions or periodic signals, i.e. those functions or signals which exhibit a regularly repeating pattern. The requirement piecewise continuous means that it is continuous on all but the finite number of points at which the left limit and the right limit exist. Since many of the phenomena studied in engineering and science are periodic in time or in space in their nature, it is understandable why periodic functions and their analyses are gaining importance. For instance, even the position of the Earth relative to the Sun is a periodic function of time with the period, which is approximately one calendar year.

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## 2. Fourier series with applications

In the 19th century, **Jean Baptiste Joseph Fourier** first came up with the idea that it is possible to write any periodic function as an infinite sum of sine and cosine terms of increasing frequency, [1]. It means that any space or time varying data can be represented in a different domain called the frequency space. In other words, Fourier series concerns periodic functions $f(x)$ of period $2L$ and they are of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

with coefficients, called Fourier coefficients of $f(x)$, given by Euler formulas

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

where $n = 1,2, ....$[2]. Dirichlet's conditions on $f(x)$ were established as a simple criteria, which determine when a Fourier series converges. The decomposition of any periodic function into the sum of the sine and the cosine functions, namely Fourier series, is called **Fourier** or **spectral analysis**, and coefficients $a_n$ and $b_n$ represent the spectrum of the function. The sine and the cosine functions (a shifted sine function) for the decomposition are called the basic functions or harmonics and index $n$ specifying the number of cycles of the sinusoid that fit within one period $2L$ and determinates the frequency of harmonics (number of oscillations in one second). This idea was presented as an important tool while Fourier developed the theory of heat conduction in his main work *Analytic Theory of Heat.* Like many other scientists, he had to struggle to get his idea accepted. His idea was elaborated further by Euler and by Daniel Bernoull, and by Dirichlet and by Riemann a bit later.
At the faculties of engineering, undergraduate university students take a class on the Fourier series in the first year of studies with the objective of finding the Fourier approximation for various squares, sawtooths, and other waveforms that occur in electronics. Mathematics for the graduate university study programme includes the Fourier method to find solutions to a certain class of linear partial differential equations (PDE), such as the heat, wave, and Laplace's equation.
Today, Fourier's idea plays a crucial role in almost all areas of modern physics, engineering, and other sciences, such as in signal processing and communications, electrical engineering, spectroscopy, crystallography, etc., where the space or time varying quantity can be a speech or an audio signal with a sound amplitude that varies in time, a temperature reading at different hours of the day, stock price changes over days, etc.

### 2.1 Spectral analysis on the phenomenon of sound

The application of Fourier series can be introduced already in high schools on the phenomenon of sound in a very simplified way. Sound reaches our ear as a longitudinal pressure wave; more precisely, as a periodic compression and rarefaction of air. When a string instrument is played, the string and surrounding air vibrate at a set of frequencies to produce a wave with a composition of harmonics. The complex sound wave, such as a note produced by instrument, is the combination of many components. It is not a single sinusoid of a single frequency, but a summation of many sinusoids with different frequencies, each contributing with a different amount. The Fourier analysis enables us to analyze constituent components in this summation, i.e. the exact composition of harmonics that determine the timbre or quality of sound that is heard, and it differentiates the sound of two different instruments even when they are playing the same note. If there is only a single harmonic sounding out in the composition, then the sound is rather pure sounding. On the other hand, if there are a variety of frequencies sounding out in the composition, then the timbre of the sound is rather rich in quality.
The following illustration shows the various sounds and their spectrums, which will enable a better understanding of the principle of spectral analysis. A pure tone is shown in the top left corner as a simple sinusoidal oscillation of a fixed frequency. In the middle is a complex sound that consists of a basic pure tone and its harmonics. The noise in the top right corner is a non-periodic oscillations of

arbitrary amplitude and frequency. The result of the spectral analysis of those sounds is shown in the lower part of Fig.1. It should be noted that a pure and complex tone has a discrete spectrum, where the certain frequency is shown on the abscissa, and the amplitude of each component is shown on the ordinate.

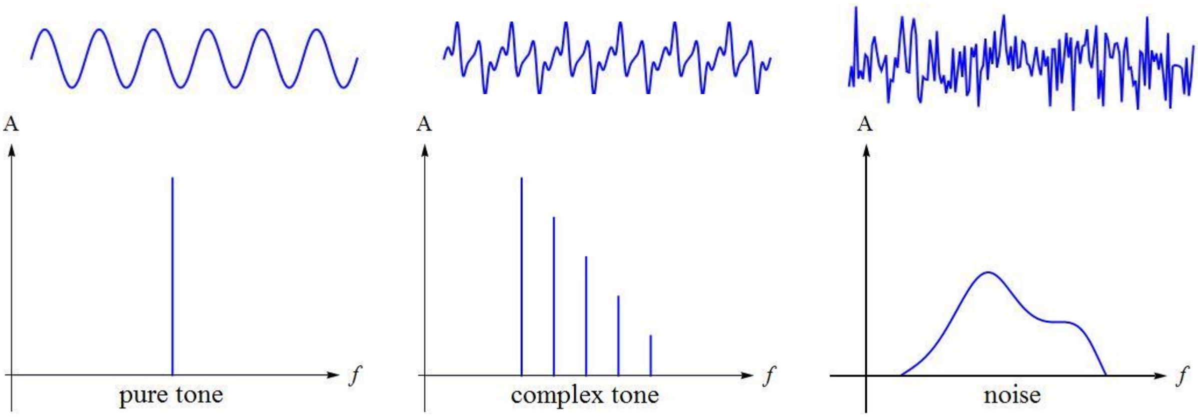


Fig.1 Different sounds and their spectrum

A pure tone vibrates with only one certain frequency, while a complex tone is distinguished by a fundamental frequency with largest amplitude and higher frequencies whose amplitude usually regularly decrease. A spectrum for the same note produced by different instruments would have different frequencies. On the other hand, it should be n oted that the noise has a continuous spectrum. This arises from the fact that its components are very densely distributed and that they cannot be individually selected. Therefore, from the spectrum one can fairly conclude the nature of the analyzed sound. The spectral representation in the frequency domain gives a much clearer explanation of why the instruments sound differently than the time domain signal. One can see how the components differ and by how much. The spectral representation also offers many opportunities for varieties of signal processing, which would not be so easy to do in the time domain.

## 2. Fourier transform with applications

Ideas and techniques of the Fourier series can be extend to nonperiodic functions $f(x)$, as a limiting case of periodic functions as the period tends to infinity, and defined on the entire real line, which leads to the Fourier integral

$$f(x) = \int_0^\infty [A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]d\lambda$$

where

$$A(\lambda) = \frac{1}{L}\int_{-\infty}^{\infty} f(t)\cos(\lambda t)\,dt,$$

$$B(\lambda) = \frac{1}{L}\int_{-\infty}^{\infty} f(t)\sin(\lambda t)\,dt.$$

Images in the spatial domain can be easily transformed into the frequency domain by the Fourier transform, which is the complex form of the Fourier integral. [4] The term frequency often describes the characteristics of some periodic motion as some variation in time in physics, but at the same time, it can be interpreted as a function of spatial coordinates in computer vision that has an impact on the variation in brightness or color across the image. [3] The Fourier transform of a continuous and integrable function of a real variable $f(x)$ is defined by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx,$$

where $\omega$ is called the frequency variable. When a function $f(x)$ is represented in the frequency space as $\hat{f}(\omega)$, then $\hat{f}(0)$ is the lowest frequency term, which is the sinusoid making the major contribution to the image, i.e. the average brightness.

Given the integrable $\hat{f}(\omega)$, one can go backward to get $f(x)$ by using the inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega.$$

The frequency domain has advantages not only for distinguishing sounds, but for image processing and filtering, because some operations and measurements can be done better in the frequency space than in the spatial domain. For example, most of the noise and improved visibility of the image can be eliminated by applying the inverse Fourier transform to retrieve the original image. The undergraduate university students of electrical engineering take a class on the Fourier transform and some of its application in the second year of study. The Fourier transform is also beneficial in solving ordinary differential equations (ODE), because it can transform them into algebraic equations which may be easier to solve. The second order linear homogeneous and nonhomogeneous ODE with constant coefficients arises in mechanical or electrical engineering in connection with vibrations and resonance or electrical circuit, [5].

The numerical implementation of the Fourier transform is called discrete Fourier transform (DFT). Usually, in practice the algorithm known under the name fast Fourier transform (FFT) is used for evaluating DFT efficiently. FFT is very useful algorithm for analyzing the human voice and speech, especially in audio forensics.

## 3. Conclusion

Fourier's legacy will certainly not be forgotten, because today the Fourier analysis arises in various areas, from analyzing the vibrational signal produced by an earthquake, preventing unwanted vibrations in cars, improving radio reception, voice recognition up to compressing digital photographs and data. Mathematics teachers should share these findings with their students. Moreover, the processing of mathematical topics like the Fourier analysis should highlight possible applications and impacts on other areas of science as much as possible. This not only encourages the motivation of students, but also affects their competences in future professions.

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