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# Using Folding Back as a Pedagogical Design Tool Under the Lens of the Van Hiele Model With Preservice Teachers

**Alberto Arnal-Bailera[1], Víctor Manero[2]**

University of Zaragoza, Facultad de Educación, Zaragoza, Spain[1]
University of Zaragoza, Facultad de Ciencias Humanas y de la Educación, Huesca, Spain[2]

## Abstract

*The well-known Van Hiele model of geometric reasoning establishes five levels of development, from level 1 (visual) to level 5 (rigor). On the other hand, the Pirie/Kieren model describes the importance of folding back processes in the learning of mathematics. This paper presents an activity implemented with mathematics teachers in training, which promotes folding back processes through the use of Van Hiele's level 5 and 4 tasks. Our results show how working with Van Hiele level 5 tasks favors reflection on similar level 4 tasks, leading to a greater depth in the reasoning of the latter. We consider that this type of activities can be especially useful in the case of students at a certain level showing some weaknesses typical of previous levels.*

**Keywords:** *Van Hiele model, Geometric Reasoning, folding back, Preservice Mathematics Teachers*

## 1. Introduction and objectives

The Van Hiele model has been proved to be very useful in relation to the teaching and learning of Geometry [1, 2]. This theoretical framework describes in some detail the competences displayed by students through their progress in the Geometrical thinking. When it comes to a particular concept, [3] formulated a model showing the non-linear way of its learning process, with different progress and setbacks, in particular folding back processes are of great interest in teacher training since it requires that the preservice teacher re-visits previous ideas of a mathematical concept in order to prepare its teaching activity.

In this study, we are working with a group of mathematics preservice teachers. All of them hold a Mathematics or Physics degree and are enrolled in a Master's degree program to become Secondary school Mathematics teachers. In this regard, our prospective secondary school teachers would show a high level of mathematical training which contrasts with the absence of education courses in their undergraduate studies. However, previous studies have shown some weaknesses in the development of their Geometrical thinking [4].

We have observed that, sometimes, our prospective mathematics teachers think that secondary school activities are too easy and they do not even need to solve them to make an in-depth analysis. Thus, they are more receptive to solving more difficult activities, probably due to their mathematical over-qualification. Thus, we have considered proposing to our students an activity consisting in the resolution of high-level activities (in Van Hiele's terms) in order to promote improvement at medium-high levels via the emergence folding back processes. In subsequent activities (out of the purpose of this work), they will be asked to design similar upper-middle level activities to promote improvement at lower levels.

Our aim is to analyse how an activity classified as level 5 (in Van Hiele's terms) favours the acquisition (or the improvement) of level 4 strategies or competences. In particular, this activity would possibly make students review their primitive knowing [3] of the proof of the sum of the internal angles of a triangle in the Euclidean metric (VH4) and rethink how to give arguments about it. The element in this reflection that invokes previous knowledge would be the work with a similar activity but using a different metric (Taxicab) and its relation to the Euclidean metric [5, 6].

## 2. Theoretical framework

### 2.1. Folding back

The Pirie-Kieren theory [3] describes eight possible levels of a person's understanding of a concept (see Figure 1). These levels are represented as nested circles, this nesting that the evolution of understanding is not necessarily linear nor unidirectional. Each level or layer contains the inner ones, i.e., it presupposes the understanding described in those levels. The eight levels range from the most localised on the inside to the most general on the outside. The model makes it possible to follow the evolution of a student's understanding of a mathematical concept, as shown in Figure 1 for a hypothetical path of the evolution of the understanding of a mathematical concept for a particular student over a given period of time.

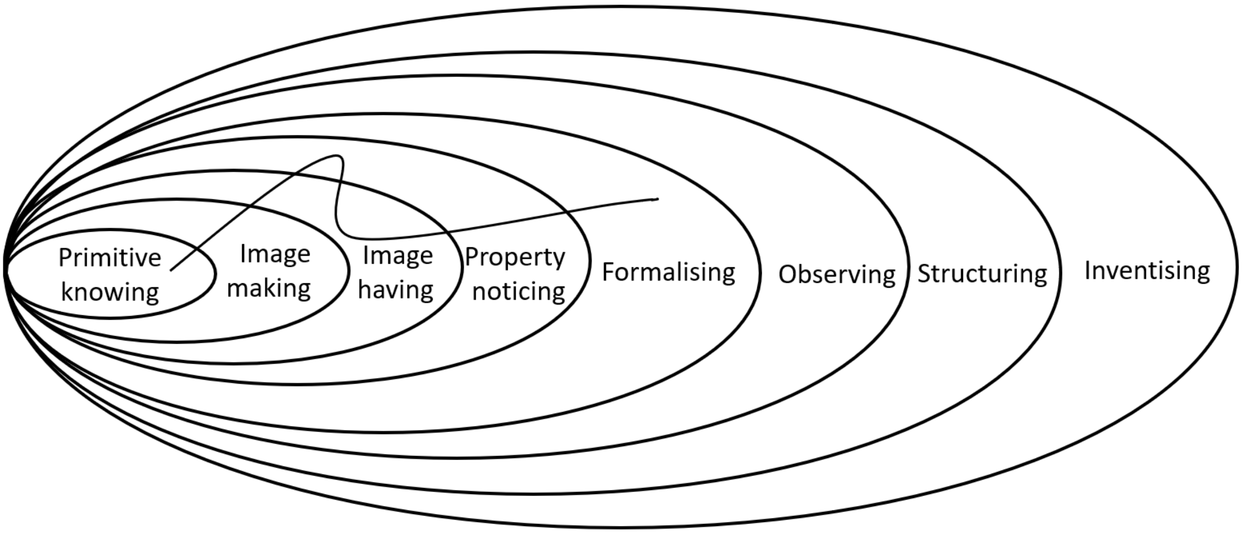



Fig.1. Layers of understanding and an instance of folding back in a hypothetical path of growth of understanding [3]

The eight levels are summarised below:
Primitive knowing: all the learner's prior knowledge except that relating to the particular topic being worked on. It is the starting point for the growth of understanding in Mathematics.
Image making: at this level, prior knowledge is used in new ways to generate particular representations (not necessarily visual) of the mathematical idea are constructed in order to approach the idea of the concept.
Image having: at this level, mental plans are made to solve activities without the need for concrete examples.
Property noticing: students examine their own images of the concept to articulate properties and connections.
Formalising: at this level the learner is able to generalise properties and work with the concept as a formal object.
Observing: the learner reflects on his/her own observations, is aware of the internal relationships that exist and organises them as a theory.
Structuring: the learner is aware of the relation between a collection of theorems and asks for formal justifications of the statements.
Inventising. The learner is able to go beyond initial ideas about a concept and create new questions that could lead to an entirely new concept.
The layers or levels described are the learning part of the model. With respect to the teaching part, [7] define "provocative", "invocative" or "validating" activities. A provocative intervention is one that takes the learner to a more general level of understanding. An invocative intervention is one that brings the learner back to an internal, more concrete level of understanding. A validating intervention maintains the current level of understanding. Note that what determines the type of activity is not the teacher's purpose but the learner's response. Thus, the same activity could possibly be of two different types depending, at least, on the learner's primitive knowing.
The act of returning to an inner layer, possibly to re-work previous ideas of a mathematical concept, is termed ‘folding back’ within the Pirie–Kieren Theory, and it is on this phenomenon that this research focuses. Folding back has been defined as
A person functioning at an outer level of understanding when challenged may invoke or fold back to inner, perhaps more specific local or intuitive understandings. This returned to inner level activity is not the same as the original activity at that level. It is now stimulated and guided by outer level knowing. The metaphor of folding back is intended to carry with it notions of superimposing ones current understanding on an earlier understanding, and the idea that understanding is somehow ‘thicker’ when inner levels are revisited. This folding back allows for the reconstruction and elaboration of inner level understanding to support and lead to new outer level understanding. [8] (p. 172)

[3] develop this definition explain that, after the folding back, the student is expected to broaden or deepen his or her current understanding by reorganising his or her previous ideas about the concept, even creating new images if necessary. This reorganisation of ideas causes the internal levels to change as the mathematical activity progresses, becoming more robust or more complete. This improvement at the innermost levels will serve to support and extend understanding at the outer levels on which further work is to be done. This inner level action is part of a recursive reconstruction of knowledge, necessary to further build outer level knowing. Different students will move in different ways and at different speeds through the levels, folding back again and again to enable them to build broader, but also more sophisticated or deeper understanding.

[9] developed a framework for folding back as an analytical observation tool. The framework identifies some categories and sub-categories that describe key aspects of folding back:

The source, which answers the question about who is prompting the shift from a layer to an inner one: an intervention by the teacher, by another learner, the curriculum material or the learner him/herself (self-invocation).

The intention, which answers the question of the source's willingness to cause the folding back: intentional or un-intentional.

The form, answers the question of what kind of actions the learner takes in reaction to the source intervention: working at an inner layer using existing understanding, folding back to collect at an inner layer, moving out of the topic or causing a discontinuity.

The outcome, answers the question of what the outcome of the act of folding back is and whether folding back has proved effective in promoting the growth of understanding: effective folding back (with or without external prompt) or ineffective folding back.

With a close but different point of view, [10] show that teachers use folding back as a pedagogical design tool since the teacher's actions always had in mind the creation of opportunities for emergence of folding back processes.

2.2. Van Hiele model

In the 50's of the XXth Century Pierre and Dina Van Hiele established the basis of one of the most relevant theoretical frameworks concerning teaching and learning Geometry, the Van Hiele model [11]. This theoretical framework states the existence of five different levels of geometric reasoning where the different geometric concepts are used and understood differently [12, 1, 13, 2].

The main characteristics of these levels consist of its sequential and hierarchical nature, meaning that they are acquired in a specific order throughout the learning process. The essential features of every level can be summarized as follows: students at level 1 (visualization) recognize geometric figures by their appearance and as a whole. Descriptions of figures are made using physical characteristics or by comparison with everyday objects by means of a nonmathematical language. Level 2 (analysis) is characterized by the ability to distinguish elements and properties of figures, which allows them to deal with mathematical descriptions of geometric concepts. The reasoning at level 3 (informal deduction) uses logical deductions in the first place, which enable students to interrelate properties of geometric figures. Thus, these students can understand logical classifications of families of figures, construct definitions as sets of necessary and sufficient conditions and provide some general arguments to justify the validity of a mathematical statement. Students at level 4 (formal deduction) can produce formal proofs and deal with equivalent definitions of a concept. Finally, people at level 5 (rigor) can compare systems based on different axioms.

It should be noticed that almost all the related literature has been focused on the development and study of the first four levels. This lack of studies on the fifth Van Hiele level can be explained by the fact that it is not related to the contents or abilities taught in the school geometry since the reasoning on different axiomatic systems appears only at University Geometry courses. Our interest in this work is related to the interest of the study of the characteristics of the Van Hiele level 5 to design activities that promote, via folding back processes, the acquisition of lower levels.

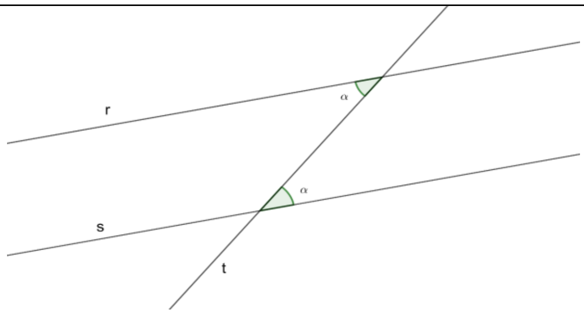
## 3. Method

Both authors teach a course on activities design in a Master's degree which is compulsory to work as a secondary school teacher. This course includes contents about van Hiele theory and the relevance of the concept of folding back in this context. The activity we present in this section has been carried out by 17 students working in groups of three. The students worked for three hours to solve a longer sequence that included this activity.

The tasks that form this activity are designed to provoke a folding back process in the students: they have to revisit a concept or process (in this case a proof) and reinforce it. Thus, we are working with an invocative activity [7]. Invocative activities are really useful when working with students who show some errors or who are not precise enough in a basic activity but are nevertheless capable to cope with a more difficult one [4]. In terms of the Van Hiele model, tasks A and B1 are to be answered by students working at Van Hiele level 4 since these tasks ask for two formal proofs. On the other hand, task B2 is to be answered by students working at Van Hiele level 5 since solving this task requires comparing two different metrics [5].

Table 1. Activity statements

TASK A. Consider the Euclidean metric. Prove that the sum of the three angles of a triangle adds up to the straight angle.
Property. Remember that, in the case of working with the Euclidean metric, if two parallel lines intersect with a secant, the alternate interior angles are equal.



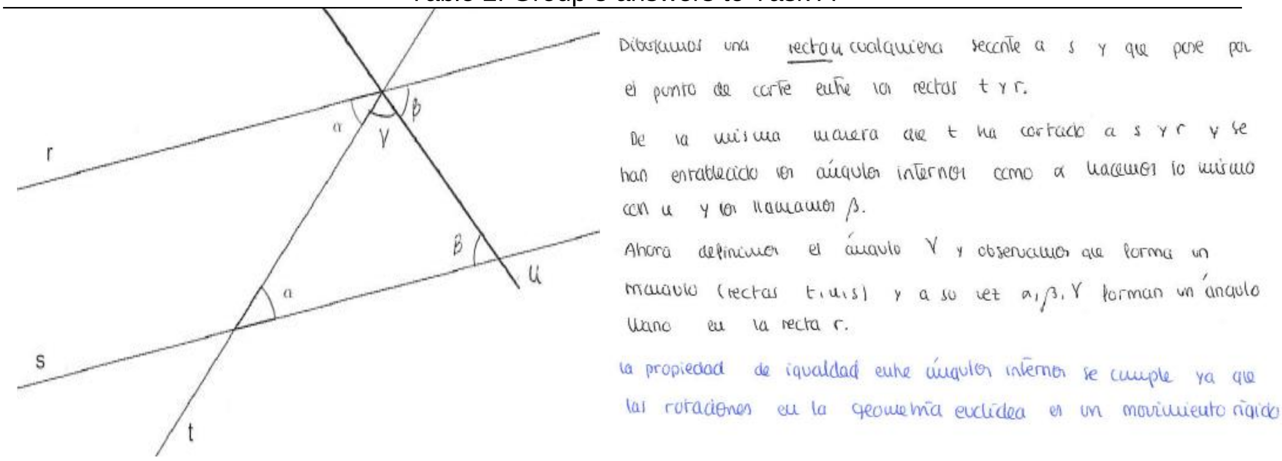

TASK B1. Consider now the Taxicab metric. Prove that the sum of the three angles of a triangle adds up to the straight angle.
TASK B2. Can we make use of the property in the case of working with the Taxicab metric, why?
Note: If you wish, you may modify your answer to Task A in the copy you have received.

Students follow a process to complete the activities: they first have to answer Task A and hand it in to the teacher. At this point, they receive Tasks B1 and B2 together with a copy of Task A in case they want to modify their response to it. In this way, the researchers have two copies of Task A: the original and one with some comments or corrections.
After collecting the answers, the researchers analysed them looking for signs to identify the four categories explained in [9]: source, intention, form and outcome. The units of analyse are the different sentences, comments and drawings written by the students in each of the tasks.

## 4. Data and Transcripts

We present and comment now a transcription from the work of group 3. This group was formed by three students, all of them holding a degree in Mathematics.
In Table 2, we can read the answers to task A (black): they prove that the sum of the three angles of a triangle adds up to the straight angle using the given property assuming it is true without any comment.

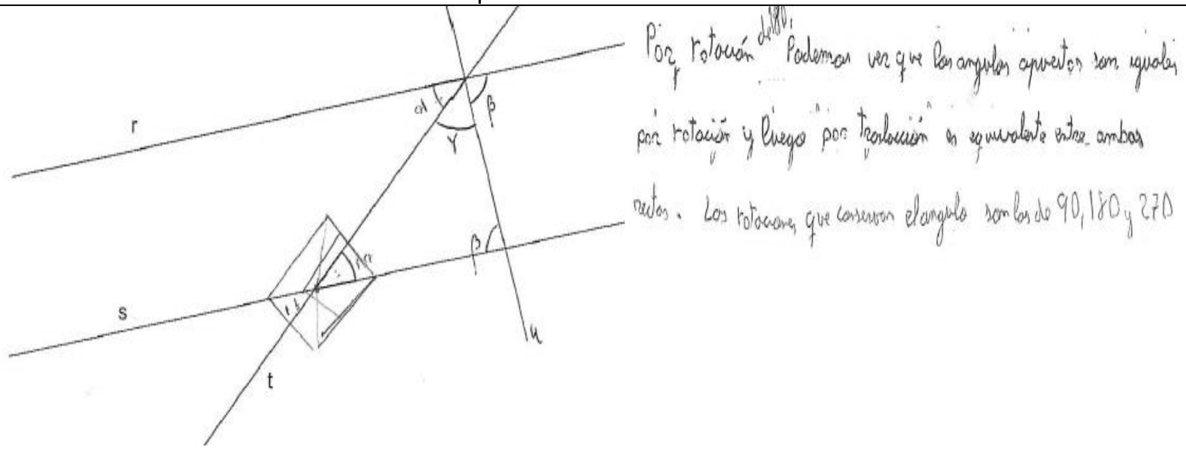
Table 2: Group 3 answers to Task A




*Drawing any secant line (u) to s and passing through the intersection point of t and r.*
*t intersects s and r forming internal angles (α), the same is done with u forming angles β. We define now angle γ, observing that forms a triangle (lines t, u, s) and α, β and γ form a flat angle in line r.*
*The property of the congruence of internal angles is satisfied due to the fact that rotations are a rigid movement in Euclidean Geometry.*

In a second moment (see Table 3), they prove the same statement working with the Taxicab metric. They introduce mathematical language using “rotation” and “translation” and state which movements preserve the angles with Taxicab metric.
Finally, (see Table 2), they add the comments in blue to precise the reason of the congruence of internal angles. They transfer their work in Tasks B1 and B2 stating that in Euclidean Geometry every rotation preserves the angles.

Table 3. Group 3 answers to Tasks B1 and B2




*Rotating 180º, we observe that vertical angles are congruent. After, by translation is equivalent in both lines. Rotations that preserve angles are 90, 180 and 270.*

It is clear that students in Group 3 have increased the mathematical rigor of their answer to Task A thanks to their work in Tasks B1 and B2. In particular, after solving B1 and B2 they added a sentence about why the given property is satisfied with Euclidean metric, showing a deeper understanding and a reorganisation of their previous ideas. Thus, the added sentence is the result of a folding back process.
A complete answer to the activity demands to structure a set of two Theorems (the sum of the internal angles of a triangle is 180º in Euclidean or in Taxicab metric).
When students were asked to compare the answer in Taxicab metric and Euclidean metric, they had to find their mutual relationships by constructing a structure consisting of these two metrics connected by the fact that they both preserve angles by 180º rotations [14]. The last task (B2) and the Note prompted the necessity of including more reasons to prove it leading to a better formalising of the proof of the Theorem in Euclidean metric by justifying a rule (The property of the congruence of internal angles).
The source of the shift from a layer (structuring) to an inner one (formalization) has been, mainly, the curricular material. However, multiple interventions of the students fostered this process. The curricular material has been intentionally designed to cause the folding back to make students work at an inner layer. We can assure that the folding back has been effective in this group with an external prompt in the form of a sentence completing a proof process.

## 5. Conclusion

After the analysis of the answers to the activity, we have confirmed that the Van Hiele level 5 activity (studying how the two metrics are related) may favour a more detailed analysis of some of the particular demonstrations in one of them, which would be a level 4 activity.
[9] showed that activities related to the introduction of different metrics were interesting for provoking folding back processes among preservice mathematics teachers. In particular, he found that unintentional peer intervention led to working at an inner layer using existing understanding.
In our case, the first source of the folding back has been the curriculum material. However, since our students were working in small groups, their comments promoted a faster folding back. The design of the intervention has been intentional, we are explicitly looking for a specific form of folding back to an inner layer to thicken the understanding of, in this case, a proof. In our case, as a result of the structuring activity, the formalization of the proof in Euclidean metric has improved by adding a justification that increases its internal logic.
It remains open to study the rest of the activities proposed to prospective teachers with the purpose of provoking other types of folding back like one of the tasks already proposed in [15] concerning the equivalence of the definitions of midset and midpoint in Euclidean and Taxicab metric. In addition, we

intend to study how working with these activities can help them to design activities based on folding back processes as well.

## 6. Acknowledgment

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