Multi-Objective Approach to Simultaneous Teaching of Thematically Related Methodological Units. Part I: Description of the method

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Abstract

This paper is devoted to a new approach to simultaneously teaching several thematic mathematical units. The combination of improper integral and function investigation thematic units is used to compile a model of consideration. The main idea is based on the Partition of Unity Method for solving multistage problems. To this end, a set of appropriate problems is developed and classified. Multistage problems arising from the fractional Laplace operator are also created. A generator of problems related to the classical Basel problem is obtained by bilinear finite element transformations. In the current decade, insufficient time has been devoted to training undergraduate students in various majors to study functions defined by integral operators. Such tasks are usually left to the training of postgraduate students. This paper provides a different perspective on this issue.

Keywords: *multi-objective approach, simultaneous teaching,*
thematically related methodological units, improper integrals, partition of unity method, multistage problem, Basel problem, problem generator

1. Introduction

Simultaneous teaching of thematically related methodological units requires a balance between online learning and in-person communications, which presents various technological, pedagogical, and organizational challenges. The effectiveness of this approach is dependent on structured teaching strategies and an interactive environment that facilitates knowledge acquisition and practical skill development [12, 15, 22]. Blended synchronous learning improves interaction but needs prior planning and adaptive methods for optimal implementation in the educational process [2, 7]. Research has shown that simultaneous learning improves cognitive processing and the efficiency of education. Active learning leads to better comprehension of the material and reduces performance disparities among students [5, 17]. Implementing simultaneous learning strategies in various disciplines has demonstrated positive effects on knowledge retention and practical application [20]. Metacognitive strategies support understanding and knowledge application, allowing for deeper content assimilation [3, 27]. Despite its advantages, cognitive fatigue remains challenging, necessitating flexible strategies for acquiring multi-objective knowledge and skills [10, 13]. Parallel teaching enables simultaneous instruction of different student groups, enhancing individual attention and teaching effectiveness. Despite its benefits, differences in teaching styles may lead to inconsistencies in content delivery and variations in student assessments. Team teaching, combined with parallel instruction, increases student engagement and fosters collaboration among educators. This requires clear organization, content alignment, and structured teaching strategies [19, 24].

Teaching improper integrals needs a balanced approach that combines theoretical and practical strategies [11, 30]. Research has shown that students struggle with understanding the concept of infinity and the algorithms for computation. Graphical representation aids in comprehension, but traditional teaching methods often fail to facilitate the learning process [16]. This requires the introduction of new pedagogical strategies that integrate analytical and visual approaches for deeper understanding and practical application [1, 23]. Todorova-Lazarova and Todorov [29] have examined online learning of improper integrals by applying the Partition of Unity Method to break down complex mathematical problems into smaller subproblems. In this model, the students have gone through a sequence of analytical steps that have supported the gradual acquisition of knowledge. Their approach involves deep recursive e-learning, where digital platforms have provided adaptability in teaching. It has explored how this approach has facilitated the study of interrelated mathematical

topics, including limits and improper integrals, through structured learning strategies and the use of electronic resources [29].

Suconota et al. have investigated the effectiveness of Design Thinking as a pedagogical strategy for teaching mathematical functions in a university context. Using a quasi-experimental design with 138 students from engineering and natural sciences, they found that students taught through Design Thinking have achieved statistically significant improvements. The most substantial progress has been observed in tasks requiring the application of mathematical models in real-world situations. More than 53% of the students have reported that project-based work within the Design Thinking methodology has improved their academic preparation [25]. Tall and Katz [26] have examined Cauchy's understanding of functions and their role in mathematical analysis. They have analyzed the influence of Cours d'Analyse on modern teaching. They have also demonstrated that, in his 1821 work, Cauchy treated functions as processes rather than merely static relations. Unlike the epsilon-delta formalism of Weierstrass, he has used infinitesimal changes, allowing for a more in-depth analysis of function behavior. They have emphasized that different interpretations of functions have shaped the way analysis is taught. Although modern approaches are based on set theory, historical perspectives have shown that intuitive methods facilitate comprehension. This suggests that Cauchy's approach remains relevant for students who struggle with the abstract formalism of contemporary analysis [26].

The paper's major contribution is the introduction of a structured method for simultaneous teaching of function investigation and improper integrals. The approach is based on the partition of unity method that allows multistage problems to be divided into finite number easy solvable subproblems. This makes the learning process more manageable and focused. The paper presents a selected list of integral type functions that can be fully analyzed without using computer tools. These functions serve as practical examples for students and support deeper understanding through analytical and graphical methods. The main goal of this investigation is to provide a rigorous and pedagogically sound methodology. It integrates the teaching of fundamental analytical tools in calculus through a synchronized learning process. The approach embeds stages of function analysis into the study of integral-type functions. In doing so, it aims to reduce cognitive fragmentation and promote deeper mathematical insight. This investigation support instructors in designing and applying a flexible teaching model as well. The method follows a recursive structure and encourages conceptual understanding. It further strengthens the student's ability to solve problems, even when they face difficulties at specific stages of the analysis.

2. Description of the method

This paper is devoted to teaching two or more thematically related methodological units simultaneously. More specifically, we consider the simultaneous teaching of investigating functions and improper integrals. The teaching approach in this article requires separate lectures to gain new knowledge on the topics of function investigation and improper integral. Exercises for each of these topics are done separately. Then, at a later stage, the two topics should be exercised simultaneously. Training in these educational units is essentially based on the Partition of Unity Method [29]. The procedure involves the following steps.

- Choice of a multistage problem on integral function that allows exact calculations for all of the above points concerning function investigation.
- Division of the main problem into a finite number of nondivisible subproblems.
- Describing all necessary definitions, properties, and theorems necessary to solve each subproblem.
- In case a student cannot solve a fixed subproblem, he should assume that the corresponding point is fulfilled and move on to the next subproblem. Failure at some level should not mean stopping work on the entire multistage problem.
- The lecturer asks the students to describe the theoretical units they have used to solve each of the individual subproblems. He requires the students to make relations between mathematical objects studied in different thematic units.
- The lecturer insists that students look for more than one solution to each subproblem.
- At the final level, the students are required to compile a division of a multistage problem.

To implement this approach, we need a large list of multistage problems that can be used for class work, independent work, and homework. This paper deals with the definition of functions that can be investigated without approximate calculations and computer technology for the purposes of mathematics education.

We focus our considerations on

$$F(x) = \int_a^{\varphi(x)} f(t)dt, \qquad a \in R. \qquad (2.1)$$

We concentrate our efforts on obtaining a set of functions of integral type (2.1) that can be investigated without computer technical support. Many of the integral-type functions considered in this paper, such as the error function, logarithmic integral, exponential integral, and Fresnel integrals, are examples of functions with nonelementary antiderivatives, in accordance with Liouville's theorem and its corollaries [9]. To this end, we obtain classes of functions that satisfy all the requirements, see Tables 1 and 2.

Table 1. Integral functions with irrational integrands

Function $f(x)$	Function $\varphi(x)$
$\displaystyle\int_0^{\varphi(x)} \frac{dt}{\sqrt{1+t^n}}, n \in N$	$x, \; x^2, \; \dfrac{1}{x^2+1}$
$\displaystyle\int_0^{\varphi(x)} \frac{t^\alpha}{\sqrt{(1-t)(1+t)}}\,dt, \quad \alpha \notin N$	$x, \sqrt{x}, \; \dfrac{1}{x+1}$
$\displaystyle\int_0^{\varphi(x)} \sqrt{1+t^{2n}}\,dt, n \in N$	$x^2, \; \dfrac{1}{x^2}, \; \ln x$
$\displaystyle\int_0^{\varphi(x)} \frac{t^\alpha}{1+t^{2\alpha+1}}\,dt, \alpha > 0$	$x, \; \dfrac{1}{x^2}, \; e^x$
$\displaystyle\int_0^{\varphi(x)} \frac{dt}{\sqrt{t^3+t}}$	$x, \; e^x, \; \ln(1+x)$

Table 2. Integral functions with transcendental integrands

Function $f(x)$	Function $\varphi(x)$	Function $f(x)$	Function $\varphi(x)$
$\displaystyle\int_0^{\varphi(x)} e^{-at^2}\,dt$	$x, \; \ln x, \sqrt{x}$	$\displaystyle\int_0^{\varphi(x)} e^{-\frac{a}{t^2}}\,dt$	$x, \ln x, \; \sqrt{x}$
$\displaystyle\int_0^{\varphi(x)} t^n e^{\frac{1}{t}}\,dt, n \in N$	$x, \; \dfrac{1}{x}, \; \ln x$	$\displaystyle\int_0^{\varphi(x)} \frac{dt}{1+e^t}$	$x, \; x^2, \; \ln x$
$\displaystyle\int_0^{\varphi(x)} \frac{\arctan t}{t}\,dt$	$x, \; x^2, \sqrt{x}, \; \ln x$	$\displaystyle\int_0^{\varphi(x)} \frac{\ln(1+t)}{t}\,dt$	$x, \; x^2, \; e^x$
$-\displaystyle\int_0^{\varphi(x)} \frac{\ln(1-t)}{t}\,dt$	$x, \; x^2, \; e^x$	$\displaystyle\int_0^{\varphi(x)} \frac{\ln t}{1-t}\,dt$	$x, \; \dfrac{1}{x}$
$\displaystyle\int_1^{\varphi(x)} \frac{\ln(1+\sqrt{t})}{t^\alpha}\,dt, \alpha > \dfrac{3}{2}$	$x, \; e^x$	$\displaystyle\int_0^{\varphi(x)} \frac{\ln t}{1+t^\alpha}\,dt, \; \alpha > \dfrac{3}{2}$	$x, \; x^2, \; e^x$
$\displaystyle\int_0^{\varphi(x)} \frac{\ln(1+t)}{1+t^2}\,dt$	$x, \; e^x$	$\displaystyle\int_0^{\varphi(x)} \ln \sin t \,dt$	$x, \; \arctan x$
$\displaystyle\int_0^{\varphi(x)} \ln \cos t \,dt$	$x, \; \arctan x$	$\displaystyle\int_0^{\varphi(x)} \sin t^n \,dt, \; n \in N \setminus \{1\}$	$x, \; \arctan x, \; \dfrac{x}{x^2+1}$
$\displaystyle\int_0^{\varphi(x)} \cos t^n \,dt, n \in N \setminus \{1\}$	$x, \arctan x, \; \dfrac{x}{x^2+1}$	$\displaystyle\int_0^{\varphi(x)} \sin^{2n+1}\left(t^2 + \dfrac{1}{t^2}\right)\,dt$	x
$\displaystyle\int_0^{\varphi(x)} \frac{1-e^t}{t}\,dt$	$x, \ln x$	$\displaystyle\int_0^{\varphi(x)} \frac{1-e^{-t}}{t}\,dt$	$x, \ln x$

The integral functions are generated with various upper bounds.
We present some examples inspired by the fractional Laplace operator. Let

$$\Gamma(x) = \int_0^\infty t^{x-1} e^t dt$$

be the real-valued gamma function, the constant $c(s)$ be defined by

$$c(s) = \frac{s2^{2s}\Gamma\left(s + \frac{1}{2}\right)}{\sqrt{\pi}\Gamma(1-s)}, \quad s \in (0,1),$$

and $P.V.$ be the Cauchy principal value.

Definition 2.1

The pointwise definition of the fractional Laplace operator $(-\Delta)^s$ [4] is as follows

$$(-\Delta)^s u(x) = c(s)P.V.\int_\mathbb{R} \frac{u(x) - u(y)}{|x-y|^{(1+2s)}}dy, \quad s \in (0,1)$$

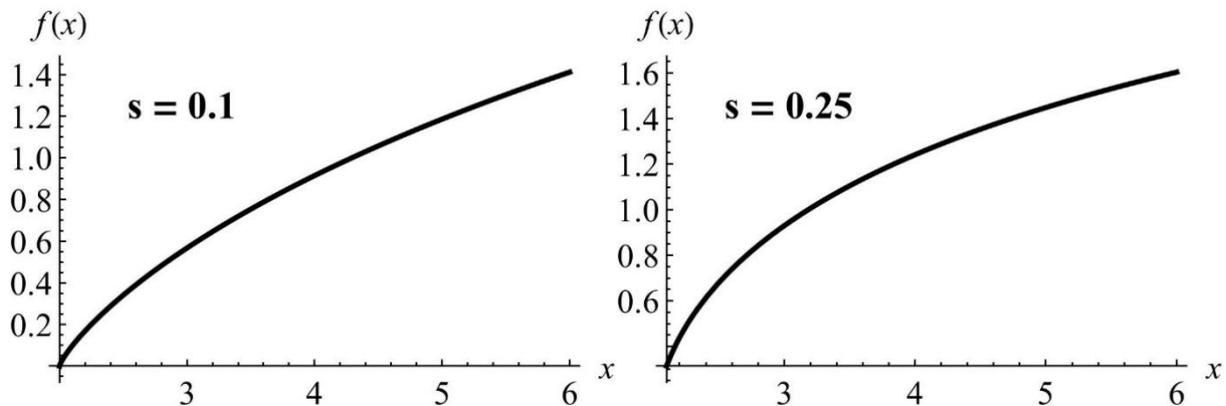So, we define a class of problems related to the investigation of the function

$$f(x) = \int_a^x \frac{u(t) - u(a)}{|t-a|^{1+2s}}dt, s \in (0,1),$$

where u is a known real-valued function and $a \in R$ is a parameter. A good choice for a representative of this class is
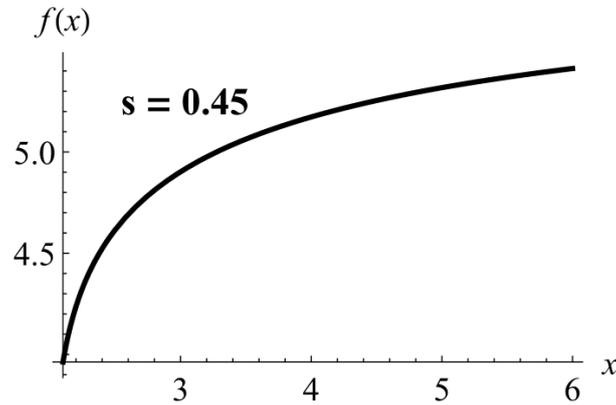
$$f(x) = \int_a^x \frac{\ln t - \ln a}{|t-a|^{1+2s}}dt, s \in (0,1) \tag{2.2}$$

with $u(x) = \ln x$. The function $f(x)$ is strictly positive, increasing and concave for all $x > a$ and $s \in \left(0, \frac{1}{2}\right)$. The graphs of the function defined by (2.2) for $a = 2$ and $s = 0.1, 0.25$ and 0.45 are presented in Fig. 1 and Fig. 2. Another appropriate example is

$$f(x) = \int_a^x \frac{e^t - e^a}{|t-a|^{1+2s}}dt, s \in (0,1).$$

Fig. 1 The grapfs of the function defined by $a = 2$ and $s = 0.1, s = 0.25$.

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$f(x)$

s = 0.45

5.0

4.5

3   4   5   6

x

Fig. 2 The grapfs of the function defined by a=2 and s=0.45.

3. The Basel problem from the educational point of view

In Part II [28], we present an implementation of the Simultaneous Teaching Method. This implementation is based on the well-known Basel problem for computing $\zeta(2)$. The calculation of the zeta function is an object of interest in [6, 18, 21].

Problem 3.1 The Basel problem

Find the exact value

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} \, . \tag{3.1}$$

The Basel problem is very attractive from an educational point of view. This problem allows for too many solutions. The most of them have been discussed by Ghosh [14]. We arrange these solutions in the following groups:
  (i)   the problem is reduced to an improper double integral;
  (ii)   the series is calculated by the Wallis integrals;
  (iii)   by Fourier analysis in Hilbert spaces;
  (iv)   based on infinite products;
  (v)   by integration in the complex plane;
  (vi)   by applying De Moivre's formula;
 (vii)   by the residuum theorem;
(viii)   based on the Weierstrass Factorization Theorem;
  (ix)   by multidimensional geometry;
  (x)   on the basis of the Fubini theorem.
To realize (i), we demonstrate a generator of problems for educational purposes. To this end, we need some preliminary definitions.

Definition 3.1

We denote an arbitrary convex polygon T by its vertices as follows
$$T = T[t_1, t_2, \ldots, t_n], \quad t_i \in R^2, \quad i = 1, 2, \ldots, n.$$
We chose the square $\hat{T} = \{\hat{t}(\hat{x}, \hat{y}) \in R^2 \mid 0 \leq \hat{x}, \hat{y} \leq 1\}$ for a reference quadrilateral with the following nodal basis functions:
$$\hat{t}_1(1,0) \mapsto \hat{\varphi}_1(\hat{t}) = \hat{x}(1 - \hat{y}), \hat{t}_2(1,1) \mapsto \hat{\varphi}_2(\hat{t}) = \hat{x}\hat{y},$$
$$\hat{t}_3(0,1) \mapsto \hat{\varphi}_3(\hat{t}) = \hat{y}(1 - \hat{x}), \hat{t}_4(0,0) \mapsto \hat{\varphi}_4(\hat{t}) = (1 - \hat{x})(1 - \hat{y}).$$
Every quadrilateral $T[t_1, t_2, t_3, t_4]$ can be obtained from the reference square by the bilinear transformation
$$T = F_T(\hat{T}), F_T(\hat{t}) = \sum_{i=1}^{4} t_i \, \hat{\varphi}_i(\hat{t}).$$
Here, we emphasize that the map F_T is linear when T is a rectangle.

In [14] the Basel problem (3.1) has been reduced to:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \int_{\hat{T}} \hat{f}(\hat{t}) d\hat{t}, \ \ \hat{f}(\hat{t}) = \frac{1}{1 - \hat{x}\hat{y}},$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 2 \int_{K} \frac{dxdy}{1 - x^2 + y^2}, \ K = \left[(1,0), \left(\frac{1}{2}, \frac{1}{2}\right), (0,0), \left(\frac{1}{2}, -\frac{1}{2}\right)\right],$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{3} \int_{W} \frac{dxdy}{1 + 2xy + y^2}, \ W = \{w(x,y) \in \mathbf{R}^2 \mid -1 \leq x, y \leq 1\},$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \int_{\hat{T}} \frac{d\hat{t}}{1 - \hat{x}^2 \hat{y}^2}.$$

We can generate new problems related to the calculation of $\zeta(2)$ by choosing a quadrilateral T and

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \int_{T} f(t) \left| J_{F_T^{-1}}(t) \right| dt, \ f = \hat{f} \circ F_T^{-1}. \tag{3.2}$$

Example 1.

Let the vertices of Ω be: $\omega_1 = (1,0), \ \ \omega_2 = (0,1), \ \ \omega_3 = (-1,0), \ \ \omega_4 = (0,-1)$.
Then by (3.2), we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 2 \int_{\Omega} \frac{dxdy}{3 + x^2 - 2y - y^2}.$$

Since Ω is a square the map $F_T : \hat{T} \to$ is linear and F_T^{-1} is also linear.

4. Conclusion

The main goal of this paper is to propose a multi-objective method for simultaneous teaching of function analysis and improper integrals. This is obtained by integrating separate stages of function analysis into the study of integral type functions. The method is based on the partition of unity and uses multistage problems divided into simple analytical steps. The functions used are of integral types and can be entirely analyzed without numerical analysis. A selected list of such functions is presented and serves as a basis for both theoretical analysis and teaching perspectives. This supports precise reasoning and structured learning. The structure of the method helps to reduce fragmentation and fosters conceptual insight. Students apply theoretical knowledge across different units. The approach encourages connections between topics. It also provides lecturers with a flexible and scalable model. The subsequent part will present the implementation of the method through detailed analysis of representative functions and classroom applications.

REFERENCES

[1] Abu-Ghuwaleh, M., Saadeh, R., Qazza, A. A Novel Approach in Solving Improper Integrals. Axioms 2022, 11, 572.

[2] Ahmad, Musa; Al-kilani, Ghazi; Al-Qatami, Mohammad (2023). The Reality of Simultaneous Teaching in Light of the COVID-19 Pandemic from the Viewpoint of Students at the College of Sports Sciences, The University of Jordan, "Jordan Journal of Applied Science-Humanities" Series: Vol. 34: Iss. 1, Article 1.

[3] Ares Ferreirós, M., Alfonso Gil, S., Rodríguez Enríquez, M., Conde Rodríguez, Á., Deaño Deaño, M. (2024). Improvement in simultaneous processing through metacognitive instruction. Frontiers in Education, 9, 1346739.

[4] B. Bahr, M. Faustmann, J. M. Melenk, An implementation of hp-FEM for the fractional Laplacian, Computers & Mathematics with Applications, Volume 176, 2024, Pages 324-348.

[5] Boris Babenko, Piotr Dollár, Zhuowen Tu, Serge Belongie. Simultaneous Learning and Alignment: Multi-Instance and Multi-Pose Learning. Workshop on Faces in 'Real-Life' Images: Detection, Alignment, and Recognition, Erik Learned-Miller and Andras Ferencz and Frédéric Jurie, Oct 2008, Marseille, France.

[6]   Jonathan M. Borwein, David M. Bradley, Richard E. Crandall, Computational strategies for the Riemann zeta function, Journal of Computational and Applied Mathematics, Volume 121, Issues 1-2, 2000, Pages 247-296

[7]   Bower, M., Kenney, J., Dalgarno, B., Lee, M., Kennedy, G. (2013). Blended synchronous learning: Patterns and principles for simultaneously engaging co-located and distributed learners. Proceedings of the ASCILITE Conference 2013.

[8]   R. Chapman, Evaluating $\zeta(2)$, Preprint,

[9]   Chaudhary, M.K., Yadav, D. K. (2024). Strong Liouville's Theorem Based Conjecture on Nonelemenrary Integrals. IOSR Journal of Mathematics, 20(1), 35-44.

[10]   Cleotilde Gonzalez, Brad Best, Alice F. Healy, James A. Kole, Lyle E. Bourne, A cognitive modeling account of simultaneous learning and fatigue effects, Cognitive Systems Research, Volume 12, Issue 1, 2011, Pages 19-32, ISSN 1389-0417.

[11]   Damian Mojica, A. (2020). Understanding of University Students About Improper Integral. International Journal of Research in Education Methodology, 11, 36-54.

[12]   Ditchev, D., Z. Petrova. On the simultaneous teaching of mathematics and new programming language in master degree. AIP Conference Proceedings, 2321(1), 030005.

[13]   Ella Miron-Spektor, Gerard Beenen, Motivating creativity: The effects of sequential and simultaneous learning and performance achievement goals on product novelty and usefulness, Organizational Behavior and Human Decision Processes, Volume 127, 2015, Pages 53-65, ISSN 0749-5978

[14]   S. Ghosh, The Basel Problem, arXiv:2010.03953.

[15]   Golshan, Behrooz; Kurti, Erdelina. (2017). Pedagogical Dilemmas in the Simultaneous Education. Dilemmas 2015 Papers from the 18th Annual International Conference Dilemmas for Human Services: Organizing, Designing and Managing. 10.15626/dirc.2015.06.

[16]   González-Martín, A. S., Camacho, M. (2004). Legitimization of the Graphic Register in Problem Solving at the Undergraduate Level: The Case of the Improper Integral. International Group for the Psychology of Mathematics Education.

[17]   A.I. Ibrahim, N. Sulaiman, I. Ali, (2022) Simultaneous multidimensional impacts of active learning revealed in a first implementation in the MENA region, Proc. Natl. Acad. Sci. U.S.A. 119 (47) e2108666119,

[18]   Y. Jerby, An approximate functional equation for the Riemann zeta function with exponentially decaying error, Journal of Approximation Theory, Volume 265, 2021, 105551.

[19]   Karjanto, N., Yong, S. T. (2018). Some pros and cons of implementing parallel and block teachings for mathematics modules. arXiv:1810.07365v1.

[20]   Masuda, A. D., Locke, E. A., Williams, K. J. (2014). The effects of simultaneous learning and performance goals on performance: An inductive exploration. Journal of Cognitive Psychology, 27(1), 37-52.

[21]   M. Milgram, An Integral Equation for Riemann's Zeta Function and Its Approximate Solution, Abstract and Applied Analysis, Volume 2020, Issue 1/ 1832982.

[22]   Popov, O. (2009). Teachers' and students' experiences of simultaneous teaching in an international distance and on-campus master's programme in engineering. The International Review of Research in Open and Distributed Learning, 10(3).

[23]   Santos, B. M. (2016). Ideas for best teaching integrals: We're teaching wrongly and how to do it right. ArXiv.

[24]   Simons, M., Baeten, M., Vanhees, C. (2020). Team Teaching During Field Experiences in Teacher Education: Investigating Student Teachers' Experiences With Parallel and Sequential Teaching. Journal of Teacher Education, 71(1), 24-40.

[25]   Suconota, E., Quizphe, R. (2024). Enhancing mathematical function understanding in university students: A comparative study of design thinking vs. Traditional teaching methods. Frontiers in Education, 9, 1364642.

[26]   Tall, D., Katz, M. A cognitive analysis of Cauchy's conceptions of function, continuity, limit and infinitesimal, with implications for teaching the calculus. Educ Stud Math 86, 97-124 (2014).

[27]   Tekin Iftar, E., Olcay Gül, S. (2017). Increasing instructional efficiency when using simultaneous prompting procedure in teaching academic skills to students with autism spectrum disorders. International Electronic Journal of Elementary Education, 9(2), 451-472.

[28]   Todorov, T.D., Todorova-Lazarova, V.P., Multi-Objective Approach to Simultaneous Teaching of Thematically Related Methodological Units. Part I: Description of the method, In Proceedings of the International Conference New Perspectives in Science Education, Florence, Italy, 2026

[29] Todorova-Lazarova, V.P.; Todorov, T.D., Some Aspects of Teaching Improper Integrals in an Electronic Environment, Mathematics 2024, 12, 2732.

[30] Zolt, H., Wrightsman, E., Ford, L., and Patterson, C. L. (2022). Believing in Infinity: Exploring Students' Conceptions of Improper Integrals. PRIMUS, 33(5), 502-516.