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# Multi-Objective Approach to Simultaneous Teaching of Thematically Related Methodological Units. Part II: Implementation

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**Abstract**

*This paper continues the study devoted to a new approach to simultaneously teaching several thematic mathematical units. The combination of improper integral and function investigation thematic units is used again to compile a model of consideration. The main idea remains based on the Partition of Unity Method for solving multistage problems. In this part, the focus is on the implementation of the method through a detailed analysis of a representative function. A multistage problem is examined step by step, including domain, limits, asymptotic behavior, monotonicity, extrema, concavity, and graphical representation. Intermediate results are obtained by applying estimation techniques and analytical transformations. The results demonstrate how the proposed methodology can be applied in practice and how it supports theoretical reasoning, structured learning, and teaching practice.*

**Keywords:** *multi-objective approach, simultaneous teaching, thematically related methodological units, improper integrals, partition of unity method, multistage problem*

## 1. Introduction

In Part I [6], we introduced a multi-objective method for the simultaneous teaching of function analysis and improper integrals. The method is based on the partition of unity and uses multistage problems divided into simple analytical steps. Its main goal is to integrate separate stages of function analysis into the study of integral type functions, in order to reduce fragmentation and to foster conceptual insight.

In this part, we focus on the implementation of the method. The approach is demonstrated through a detailed multistage analysis of a representative integral-type function. The stages of the analysis include domain, limits, continuity, asymptotic behavior, monotonicity, concavity, and graphical representation. The aim is to show how the structured approach can be applied in practice and how it supports both theoretical reasoning and teaching practice.

## 2. Implementation

We demonstrate a multistage problem, which is suitable for the simultaneous teaching of improper integrals and function investigation. The various stages in the analysis of the function

$$f(x) = \int_0^x \frac{\ln(1+t)}{t} dt$$

are presented as follows.

### *2.1 Domain of the function*

We denote the integrand by

$$g(x) = \frac{\ln(1+x)}{x}.$$

Since

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

the function $g(x)$ is bounded at zero and $\nexists g(0)$. The domains of both functions are given by:

$$D_g = (-1,0) \cup (0,+\infty), \quad D_f = [-1,0) \cup (0,+\infty).$$

### 2.2 Limits at the boundary points of the domain

Let $x > 0$ then $1 + x > 1 \Rightarrow \ln(1+x) > 0$ and $g(x) > 0$. If $x \in (-1,0)$ then $1 + x < 1 \Rightarrow ln(1+x) < 0$, which again leads to $g(x) > 0$. Therefore,

$$g(x) = \frac{\ln(1+x)}{x} > 0, \quad \forall x \in Dg.$$

The calculation of

$$f(-1) = \lim_{x \to -1^+} f(x)$$

requires an evaluation of an improper integral

$$f(-1) = \lim_{x \to -1^+} \int_0^x \frac{\ln(1+t)}{t}\, dt = \int_0^{-1} \frac{\ln(1+x)}{x}\, dx.$$

To obtain a positive upper bound, we perform the substitution $x = -t$. Thus, we get

$$\lim_{x \to -1^+} f(x) = -\int_0^1 g(-t) dt = \int_0^1 \frac{\ln(1-x)}{x}\, dx. \qquad (2.1)$$

Since $x \in (0,1)$, we conclude that the integrand satisfies

$$\frac{\ln(1-x)}{x} < 0.$$

In this stage, we present two alternative solutions. If the multistage problem is intended for students who have already studied the series expansions of one-variable functions and the summation of power series, then we will reduce the integral (2.1) to a Basel problem and calculate its exact value. However, if the students have not studied the power series, we will estimate the integral (2.1) and continue the study with approximate values of this integral.

#### 2.2.1 A solution for students who have not studied power series

In this subsection, we do not present an exact computation of the integral (2.1). We find upper and lower bounds for $f(-1)$. Let $\varphi(\alpha) = x^{-\alpha}$, where $x \in (0,1)$. Then

$$\varphi'(\alpha) = x^{-\alpha} \ln x (-\alpha)'_\alpha = -x^{-\alpha} \ln x > 0 \Rightarrow \varphi \nearrow.$$

Since $x^{-1}$ is bounded between $x^{-\frac{1}{2}}$ and $x^{-\frac{3}{2}}$ in the interval $(0,1)$, we obtain:

$$x^{-\frac{1}{2}} < x^{-1} < x^{-\frac{3}{2}} \Rightarrow \frac{1}{x^{\frac{1}{2}}} < \frac{1}{x} < \frac{1}{x^{\frac{3}{2}}}.$$

Multiplying both sides by $\ln(1-x)$ reverses the inequalities:

$$\frac{\ln(1-x)}{x^{\frac{3}{2}}} < \frac{\ln(1-x)}{x} < \frac{\ln(1-x)}{x^{\frac{1}{2}}} < 0.$$

By integrating over $[0,1]$ and applying a comparison test between the improper integrals, we get

$$\int_0^1 \frac{\ln(1-x)}{x^{\frac{3}{2}}}\, dx < \int_0^1 \frac{\ln(1-x)}{x}\, dx < 0.$$

To obtain a lower bound for $f(-1)$, we evaluate the integral

$$I_1 = \int_0^1 \frac{\ln(1-x)}{x^{\frac{3}{2}}}\, dx.$$

By substituting $t = \sqrt{x}$, we have

$$\int_0^1 \frac{\ln(1-x)}{x^{\frac{3}{2}}}\, dx = \int_0^1 \frac{\ln(1-t^2)}{t^3} 2t\, dt = 2\int_0^1 \frac{\ln(1-t^2)}{t^2}\, dt.$$

The integrand

$$h(t) = 2\frac{\ln(1-t^2)}{t^2}$$

has singularities at both limits of integration. To determine an antiderivative $h^{(-1)}(t)$, we solve the indefinite integral

$$2\int \frac{\ln(1-t^2)}{t^2}\, dx = 2\int \ln(1-t^2)\, d\left(-\frac{1}{t}\right) = -2\int \ln(1-t^2)\, d\frac{1}{t}.$$

The integration by parts leads to

$$h^{(-1)}(t) = -\frac{2}{t}\ln(1-t^2) + 2\ln\frac{1-t}{1+t}.$$

The calculation of $h^{(-1)}(t)$ can be considered as a separate subproblem. We continue with the computation of the main integral

$$I_1 = \int_0^1 \frac{\ln(1 - x)}{x^{\frac{3}{2}}} dx = 2\left[\ln\left(\frac{1 - t}{1 + t}\right) - \frac{\ln(1 - t^2)}{t}\right]\Bigg|_0^1$$

$$= 2\lim_{t\to 1^-}\left(\ln\frac{1-t}{1+t} - \frac{\ln(1-t^2)}{t}\right) - 2\lim_{t\to 0^+}\left(\ln\frac{1-t}{1+t} - \frac{\ln(1-t^2)}{t}\right).$$

The integral is expressed in terms of limits of the antiderivative at the boundary points of integration. The contributions of both limits when $t \to 1^-$ and $t \to 0^+$ can be considered as separate subproblems as well. Then

$$I_1 = 2\lim_{t\to 1^-}\ln(1-t) - 2\lim_{t\to 1^-}\ln(1+t) - 2\lim_{t\to 1^-}\frac{\ln(1-t^2)}{t} - 2\ln\frac{1-0^+}{1+0^+} + 2\lim_{t\to 0^+}\frac{(\ln(1-t^2))'}{(t)'}$$

$$= 2\lim_{t\to 1^-}\ln(1-t) - 2\lim_{t\to 1^-}\ln(1-t^2)^{\frac{1}{t}} - 2\lim_{t\to 1^-}\ln(1+t) - 2\ln 1 + 2\lim_{t\to 0^+}\frac{(1-t^2)'}{1-t^2}.$$

At this step, we decompose the logarithmic terms to separate dominant components into individual limits. This approach allows us to analyze the behavior of each term independently

$$I_1 = 2\left(\lim_{t\to 1^-}\ln(1-t) - \lim_{t\to 1^-}\ln(1-t^2)^{\frac{1}{t}}\right) - 2\ln 2 + 2\lim_{t\to 0^+}\frac{-2t}{1-t^2}$$

$$= 2\lim_{t\to 1^-}\left(\ln(1-t) - \ln(1-t^2)^{\frac{1}{t}}\right) - 2\ln 2 - 4\lim_{t\to 0^+}\frac{t}{1-t^2}$$

$$= 2\lim_{t\to 1^-}\ln\frac{1-t}{(1-t^2)^{\frac{1}{t}}} - 2\ln 2 = 2\lim_{t\to 1^-}\ln\frac{1-t}{(1-t)^{\frac{1}{t}}(1+t)^{\frac{1}{t}}} - \ln 4$$

$$= 2\lim_{t\to 1^-}\left(\ln\frac{1-t}{(1-t)^{\frac{1}{t}}} - \ln(1+t)^{\frac{1}{t}}\right) - \ln 4.$$

The function

$$\varphi(t) = \frac{1-t}{(1-t)^{\frac{1}{t}}}$$

is continuous for all internal points of the interval of integration. That is why we interchange the order of the execution of the limit and the logarithm

$$I_1 = 2\lim_{t\to 1^-}\ln(1-t)^{1-\frac{1}{t}} - 2\lim_{t\to 1^-}\ln(1+t)^{\frac{1}{t}} - \ln 4 = 2\lim_{t\to 1^-}\ln(1-t)^{\frac{t-1}{t}} - 2\ln(1+1^-)^{\frac{1}{1^-}} - \ln 4$$

$$= 2\ln\lim_{t\to 1^-}(1-t)^{\frac{t-1}{t}} - 2\ln 2 - \ln 4 = 2\ln\lim_{t\to 1^-}(1-t)^{\frac{t-1}{t}} - 2\ln 4.$$

So, the integral $I_1$ is reduced to

$$I_1 = \int_0^1 \frac{\ln(1-x)}{x^{\frac{3}{2}}} dx = 2\ln\lim_{t\to 1^-}(1-t)^{\frac{t-1}{t}} - \ln 16.$$

The term

$$L_1 = \lim_{t\to 1^-}(1-t)^{\frac{t-1}{t}}$$

is a limit of an indeterminate form of type $[0^0]$. We transform this indeterminate form into a lower-order one by

$$L_1 = \lim_{t\to 1^-} e^{\ln(1-t)^{\frac{t-1}{t}}} = \lim_{t\to 1^-} e^{\frac{t-1}{t}\ln(1-t)} = e^{\lim_{t\to 1^-}\frac{t-1}{t}\ln(1-t)}.$$

By denoting the exponent as $L_2$, we have $L_1 = e^{L_2}$. Since

$$L_2 = \lim_{t\to 1^-}\frac{(t-1)}{t}\ln(1-t) = \frac{\lim_{t\to 1^-}(t-1)\ln(1-t)}{\lim_{t\to 1^-} t} = \lim_{t\to 1^-}(t-1)\ln(1-t),$$

we conclude that $(t-1)\ln(1-t) \in [0.\infty]$ in the neighborhood of the point $-1$. At this level, we draw the attention of the students that L'Hospital's theorem is not yet applicable. We still need to make the following transformations

$$L_2 = \lim_{t\to 1^-}(t-1)\ln(1-t) = -\lim_{t\to 1^-}(1-t)\ln(1-t) = -\lim_{t\to 1^-}\frac{\ln(1-t)}{(1-t)^{-1}}.$$

The function

$$\frac{\ln(1-t)}{(1-t)^{-1}}$$

is an indeterminate form of the type $\left[\frac{\infty}{\infty}\right]$ in the neighborhood of the point $-1$, which is why we recommend L'Hospital's rule

$$L_2 = -\lim_{t\to 1^-}\frac{(\ln(1-t))'}{((1-t)^{-1})'} = -\lim_{t\to 1^-}\frac{\frac{(1-t)'}{1-t}}{(-1)(1-t)^{-2}(1-t)'}$$
$$= \lim_{t\to 1^-}\frac{(1-t)'}{(1-t)^{-2}(1-t)(1-t)'} = \lim_{t\to 1^-}\frac{1}{(1-t)^{-1}} = 0.$$

Since $L_2 = 0$, it follows that $L_1 = e^0 = 1$. Thus, we obtain:

$$I_1 = \int_0^1 \frac{\ln(1-x)}{x^{\frac{3}{2}}}\,dx = 2\ln L_1 - \ln 16 = -\ln 16,$$
$$I_1 = \int_0^1 \frac{\ln(1-x)}{x^{\frac{3}{2}}}\,dx = -\ln 16 < \int_0^1 \frac{\ln(1-x)}{x}\,dx$$
$$= \int_0^{-1}\frac{\ln(1+x)}{x}\,dx = f(-1) = -\int_{-1}^0 \frac{\ln(1+x)}{x}\,dx < \int_{-1}^0 0\,dx = 0.$$

This means that $-\ln 16 < f(-1) < 0$, i.e., $f(-1)$ is finite, and $x = -1$ is not a vertical asymptote. Therefore, the domain of function is $D_f = [-1,0) \cup (0,+\infty)$.
To analyze the behavior of $f(x)$ at $x = 0$, we evaluate the following limit:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\int_0^x \frac{\ln(1+t)}{t}\,dt.$$

Since $g(x) = \frac{\ln(1+x)}{x} > 0$, we estimate

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\int_0^x \frac{\ln(1+t)}{t}\,dt \le \lim_{x\to 0^+}\int_0^x \frac{t}{t}\,dt = \lim_{x\to 0^+} x = 0.$$

But $\nexists f(0)$ so, we have a point of removable discontinuity.

To establish the asymptotic behavior when $x \to +\infty$, we use the inequality

$$\frac{x}{x+1} \le \ln(1+x) \le x,\ x > 1. \qquad (2.2)$$

To obtain $f(+\infty)$, we continue with

$$f(+\infty) = \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\int_0^x \frac{\ln(1+t)}{t}\,dt = \int_0^{+\infty}\frac{\ln(1+t)}{t}\,dt$$
$$> \int_0^{+\infty}\frac{t}{1+t}\cdot\frac{1}{t}\,dt = \int_0^{+\infty}\frac{dt}{1+t} = +\infty.$$

Therefore

$$\lim_{x\to+\infty} f(x) = +\infty.$$

Estimating functions and integrals requires a variety of theoretical knowledge and represents a significant difficulty for students of various majors. This necessitates the development and use of a large number of problems involving the estimation of various integral operators.

**Intermediate results for Subsection 2.2.1**

$$\beta \approx -1.64493,\quad \beta = -\frac{\pi^2}{6},\quad -\ln 16 \approx -1.895025,$$
$$\lim_{x\to -1^+} f(x) = \beta,\quad -\ln 16 < \beta < 0,$$
$$\lim_{x\to 0} f(x) = 0,\qquad \lim_{x\to+\infty} f(x) = +\infty.$$

***2.2.2 A solution for students who have already studied power series***

The main goal in this subsection is to obtain the exact value of

$$f(-1) = \int_0^1 \frac{\ln(1-x)}{x}\,dx.$$

Expanding the function $\ln(1-x)$ in the Maclaurin power series and changing the order of summation and integration, we attain

$$\int_0^1 \frac{\ln(1-x)}{x}dx = -\int_0^1 \frac{1}{x}\sum_{k=1}^{\infty}\frac{x^k}{k}dx = -\int_0^1 \sum_{k=1}^{\infty}\frac{x^{k-1}}{k}dx$$

$$= -\sum_{k=1}^{\infty}\int_0^1 \frac{x^{k-1}}{k}dx = -\sum_{k=1}^{\infty}\frac{1}{k^2} = -\zeta(2).$$

So, we obtain the Basel problem for the Riemann zeta function $\zeta(2)$. Further developments on this problem are presented in Ghosh [4]. We treat the computation of $\zeta(2)$ as a subproblem in our considerations. The Basel problem is thoroughly discussed in [6]. To avoid calculations of consequent or double integrals we use the Maclaurin expansion of

$$\arcsin x = \sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2}\frac{x^{2k+1}}{2k+1} = \sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!}\frac{x^{2k+1}}{2k+1}, \quad |x| \le 1.$$

After that following Chapman [2] we place $t = \arcsin x$ in the latter equation. Finally, the calculation of $\zeta(2)$ is reduced to solving the Wallis integral

$$W_k = \int_0^{\frac{\pi}{2}} \sin^{2k+1} x\, dx, \quad k \in \boldsymbol{N}.$$

More detailed computational strategies and connections with the ζ-function are presented by Borwein et al. [1]. The crucial point in this solution is

$$\zeta(2) = \frac{4}{3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$$

.

***2.3 Asymptotes***

Since $\lim_{x\to -1^+} f(x) = \beta,\ -\ln 16 < \beta < 0$, and $\lim_{x\to 0} f(x) = 0$, it follows that there are no vertical asymptotes. From

$$\lim_{x\to +\infty} f(x) = +\infty \text{ and } D_f = [-1,0) \cup (0,\infty),$$

we conclude that there are no horizontal asymptotes.

Analysis of the slant asymptote. To determine whether a slant asymptote exists, we compute the slope $k$

$$k = \lim_{x\to+\infty}\frac{f(x)}{x}$$

$$= \lim_{x\to+\infty}\frac{\int_0^x \frac{\ln(1+t)}{t}dt}{x} = \lim_{x\to+\infty}\frac{\left(\int_0^x \ln\frac{1+t}{t}dt\right)'}{(x)'} = \lim_{x\to+\infty}\frac{\ln(1+x)}{x} = \lim_{x\to+\infty}\frac{(\ln(1+x))'}{(x)'} = 0.$$

Since $k = 0$, we now compute the intercept $n$

$$n = \lim_{x\to+\infty}(f(x) - kx) = \lim_{x\to+\infty} f(x) = +\infty.$$

Therefore, there are no slant asymptotes. We conclude that the graph of the function $f(x)$ does not have asymptotes.

***2.4 Analysis of symmetry***

Since the domain $D_f = [-1,0) \cup (0,+\infty)$ has no symmetry with respect to the origin the function $f(x)$ is neither even nor odd.

***2.5 Intersection points with the coordinate axes***

To determine the intersection points of the graph $\Gamma_f$ with the coordinate axes, we solve the system

$$\begin{cases} y = 0, \\ y = f(x) \end{cases} \Rightarrow \int_0^x \frac{\ln(1+t)}{t}dt = 0.$$

Since $g(x) > 0$, there is no solution. On the other hand

$$\begin{cases} x = 0, \\ y = f(x) \end{cases} \Rightarrow \int_0^0 \frac{\ln(1+t)}{t}dt.$$

Again, a solution is not found due to $\nexists g(0)$. Therefore, there are no intersections of $\Gamma_f$ with the coordinate axes.

***2.6 Monotonicity and extrema***

We have:

$$f(x) = \int_0^x \frac{\ln(1+t)}{t}\,dt, \quad f'(x) = \frac{\ln(1+x)}{x} > 0, \quad \forall x \in D_f.$$

Therefore, $x \in D_f,\ f'(x) > 0,\ f \nearrow$.

Since $f'(x) > 0$ for all $x \in D_f$, the function is strictly increasing. This means that the function has no extrema.

### *2.7 The range of the function*

Based on the results in the previous subsections, we assert that the range $V_f$ is $V_f = (\beta, 0) \cup (0, +\infty)$.

### *2.8 Convexity and inflection points*

$$f''(x) = \left(\frac{\ln(1+x)}{x}\right)' = \frac{(\ln(1+x))'\ x\ -\ x'\ \ln(1+x)}{x^2}$$

$$= \frac{\frac{(1+x)'}{1+x}\,x - \ln(1+x)}{x^2} = \frac{\frac{1}{1+x}\,x - \ln(1+x)}{x^2}.$$

From the classical inequality (2.2) follows that

$$f''(x) = \frac{\frac{x}{1+x} - \ln(1+x)}{x^2} < 0, \quad \forall\, x \in D_f.$$

Since $f''(x) < 0$ for all $x \in D_f$, the function is strictly concave in its entire domain. Therefore, there are no inflection points.

### *2.9 Table*

The results in the different stages obtained are placed in Table 1.

**Table 1.** The signs of the derivatives and the behavior of the function f(x).

| $x$ | $-1$ | | $0^-$ | $0$ | $0^+$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $y$ | $\beta$ | $\nearrow \cap \nearrow \cap \nearrow \cap$ | $0$ | $\vert$ | $0$ | $\nearrow \cap \nearrow \cap \nearrow \cap$ | $+\infty$ |
| $y'$ | | $++++$ | | $\vert$ | | $+++++$ | |
| $y''$ | | $-----$ | | $\vert$ | | $-------$ | |

### *2.10 Graph*

To plot the graph $\Gamma_f$ we use the following properties of $f(x)$ (see Figure 1):

- $f(x)$ is increasing;
- the function is concave;
- the points $A = (-1, \beta)$, $B_1(0^-, 0)$, $B_2(0^+, 0)$, $C(+\infty, +\infty)$ belong to $\Gamma_f$.

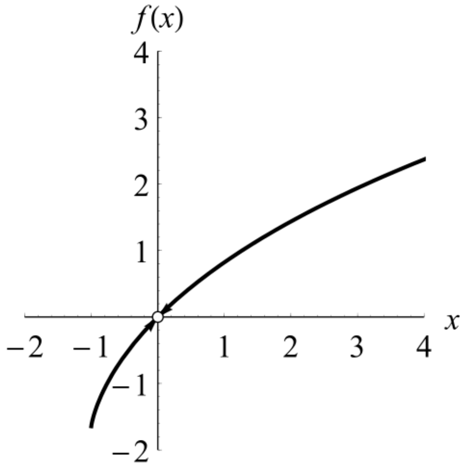


**Fig. 1.** The graph of the function $f(x)$.

## 3 Conclusion

This paper presents the implementation of a multi-objective method for the simultaneous teaching of function analysis and improper integrals. The implementation is carried out through a detailed multistage analysis of a representative integral type function. The stages of analysis are domain, limits, asymptotic behavior, monotonicity, concavity, and graphical representation. Their systematic applications demonstrate how the method can be used in practice and in a classroom setting. The methodology supports structured reasoning. It shows that it reduces the fragmentation of knowledge. It also demonstrates that it promotes deeper mathematical insight. The method is theoretically sound, and it is pedagogically effective. The implementation provides a clear framework that connects different analytical procedures. It offers both students and lecturers a flexible and scalable model. This study reveals that a rigorous method can be transformed into practical teaching tools. It shows that the approach strengthens conceptual understanding, enhances the integration of related topics, and encourages coherent mathematical learning. Similar observations regarding students' difficulties with improper integrals are also discussed in [3]. The connection between graphical representation and integral understanding is noted in González-Martín & Camacho [5], which further supports the role of structured multistage analysis. The continuity of the methodology with our previous work is emphasized in Todorov & Todorova-Lazarova [6]. The implementation demonstrates a tested pathway for bridging theory and practice in higher mathematics education.

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